HIGH FREQUENCY TRANSMISSION LINES
PREFACE

All advanced students of electrical engineering are acquainted with the elements of transmission-line theory and of its application in problems of power transmission and telephonic communication. In these applications the frequencies employed are so low that the wavelength in the line is enormously larger than the separation between the conductors, and the theory as presented in electrical engineering text-books has sufficed, in general, for their analytical solution. In the very recent applications of lines, both as interconnectors and as circuit elements, in microwave equipment, where the wavelength may be only a few centimetres and the lengths of the line elements only a fraction of a wavelength, however, this is no longer the case. Nevertheless, since the only alternative to utilization of the normal transmission-line theory is the formulation and solution of what can be extremely complex problems in electromagnetism, this theory continues to have great value—though it may give no more than a guide to probable performance—even where the physical premises of the theory are far from being satisfied. The foundations of transmission-line theory were laid many years ago by Oliver Heaviside in his books on Electromagnetic Theory and by J. R. Carson in various publications in the Bell Telephone Journal and elsewhere. No attempt has been made in this small book to give a comprehensive re-statement of the work of these pioneers, but it is hoped that sufficient has been said in Chapters II and IV to bring out the restrictions associated with the application of the theory to high-
frequency line systems. The remaining chapters are devoted to a consideration of the characteristic properties of lines and of their applications in high-frequency technique. In some places a good deal of algebraic manipulation has been left to the reader, but, being straightforward, it is felt that this is permissible.

I am indebted to Dr. L. G. H. Huxley for permission to include in Chapter VI much of the contents of a joint paper with him which has been published in the *Journal* of the I.E.E., and to the Institution for permission to make this reproduction. My thanks are also due to Mr. M. R. Gavin for assistance with the preparation of Chapter I and to Dr. L. Essen and Dr. R. L. Lamont for reading the manuscript and proofs respectively.

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March 1944
As in other branches of electrical engineering, a very important application of transmission lines at very high frequencies is that of transferring power from a generator to a load. This, however, is only one of their many uses in the very high frequency field. At frequencies of the order $10^8$-to-$10^{10}$ cycles/sec. the wavelength $\lambda$ has become so short, 3 metres–3 cms. for air dielectric lines, that certain characteristic properties of transmission lines in general can be realized with quite short lengths of line. Thus by choosing the appropriate length between one half-wavelength, $\frac{\lambda}{2}$ and zero, any reactance value between plus and minus infinity can be obtained. Moreover, the power factor of these reactances can be kept extremely low. Such lengths of line find widespread application as circuit elements in very high frequency technique when the more conventional coils and condensers fail to function satisfactorily. Furthermore, suitably chosen lengths of line are used as resonant circuits in oscillators and for the measurement of wavelength, impedance, and the permittivity and power factor of dielectric materials.

This first chapter gives a brief survey of some of these applications. The treatment is descriptive and superficial, but the principles underlying the several applications, and the restrictions attaching to them, are dealt with at length in the subsequent chapters.

HIGH FREQUENCY FEEDERS

For maximum efficiency of power transfer from an oscillator to an antenna system or other load through a
length of transmission line—or feeder, as it is called—of given attenuation, two conditions must be satisfied: (1) the impedance presented to the oscillator by the input of the feeder must be the conjugate of the internal impedance of the generator, (2) the impedance of the antenna at its point of connexion to the output end of the feeder must be equal nominally to the characteristic impedance of the latter, so that no standing waves exist along the length of the feeder.

The first of these conditions is the familiar condition for maximum power output from a generator. That the second condition is necessary will be apparent from the following considerations. For maximum efficiency of transmission through the feeder the fractional power loss in it must be a minimum. The first desirable step to this end is, of course, to employ a line of the lowest attenuation constant consistent with practical possibilities. However, the existence of a standing wave along the line means that in addition to the forward wave from oscillator to antenna there is a reflected wave travelling in the reverse direction. The power supplied to the antenna depends on the difference in amplitude of these two waves, whereas, being attenuated independently, each contributes to the total power loss in the feeder. Thus for a given power input to the feeder the losses will be least and the power output greatest when the amplitude of the reflected wave is zero, that is, when there is no standing wave. Under these circumstances the antenna is said to be 'matched' to the line, and the latter is said to be 'correctly terminated'.

The ratio of the maximum to minimum values of voltage or current along an incorrectly terminated line is known as the Standing-Wave Ratio.† It is usually possible with care to reduce this ratio to 1.1 or less. The reduction of efficiency associated with such a standing wave ratio is negligible, and indeed much higher ratios are permissible where efficiency is the only important consideration.

* See pp. 86–9. † See p. 76.

In high power systems, however, a high standing wave introduces two undesirable possibilities, (a) that the increased electric stress developed at the voltage anti-nodes may cause corona and flash-over, (b) that the high current at the current anti-nodes may cause excessive local heating.

There is a still further reason why it is advantageous to avoid the existence of standing waves. Many generators of very high frequencies consist of self-oscillators the operation of which is critically dependent on the loading conditions. Now when there is an appreciable standing wave on the attached feeder the impedance presented to the oscillator will depend on the length of the feeder in so far as the latter, along with the wavelength, determines the position of the input end of the feeder on the standing wave pattern. If this end is at a voltage node or anti-node the input impedance will be purely resistive; otherwise there will be a reactive component. Any change at the antenna or in the length of feeder is likely therefore to affect both the output and frequency of the oscillator.

The effect is particularly troublesome where the system in question is required to operate over a small frequency band. If the feeder is several wavelengths long quite a small change of frequency may exercise a considerable effect on the behaviour of the oscillator. In such cases particular care should be taken to reduce the standing wave ratio to as near unity as possible at the mean operating frequency. The position is complicated by the fact that the matching devices employed to reduce the standing wave are of necessity frequency sensitive, so that, failing continual readjustment, the range of frequency over which satisfactory operation can be achieved is strictly limited.

MATCHING DEVICES*

In order to facilitate satisfactory termination of a feeder by an aerial, special feeders have been constructed whose

* See pp. 134–42
characteristic impedances are equal to commonly occurring aerial impedances. Two examples of simple aerials fed by coaxial lines of suitable impedances are shown in Fig. 1.

![Diagram of half-wave and quarter-wave aerials fed by coaxial feeders.](image)

(a) Half-wave dipole aerial fed by a coaxial feeder of characteristic impedance between 70 and 80 ohms.
(b) Quarter-wave aerial fed by 40-ohm coaxial feeder.

Generally the terminating impedances at the ends of the feeder differ from the characteristic impedance and it is then necessary to introduce transforming devices to match the aerial and the generator to the feeder.

![Diagram of feeder coupling methods.](image)

(a) Feeder coupled to an oscillatory circuit by tapping on to the appropriate points of the circuit.
(b) As in (a) with a transmission line forming the inductive part of the oscillatory circuit.
(c) Feeder coupled magnetically.

As far as the generator (a valve oscillator) is concerned the impedance match stated in condition (1) above can be achieved by any means which varies the coupling between the oscillatory circuit and the feeder. Common methods of varying the coupling are shown in Fig. 2. In Fig. 2 (a) and (b) the feeder is tapped on to the appropriate points...
of the oscillatory circuit. In Fig. 2 (c) a small loop is attached to the end of the feeder and coupled to the inductance of the circuit. Matching devices of the type just described are usually adjusted by a process of trial and error. The impedance of a short-wave valve oscillator is usually not known with any degree of accuracy so that the adjustment for maximum power output has to be made empirically. The case of a feeder connected to any resonant circuit, e.g. the input circuit of a receiver, can be treated in a similar way.

The methods of Fig. 2 are not normally applicable at the load end of the feeder and special transforming or matching devices have been evolved for the purpose of satisfying condition (2).

One of these devices is the Quarter-Wave Transformer. It is shown later that the input impedance \( Z_g \) of a loss-free line one quarter-wavelength long which is terminated in an impedance \( Z_T \) is given by the relation

\[
Z_g Z_T = Z_0^2
\]

in which \( Z_0 \) is the characteristic impedance of the line. Thus a load of impedance \( Z_T \) may be matched to a feeder of characteristic impedance \( Z_g \) by the insertion between them of a \( \lambda/4 \) length of line of characteristic impedance equal to the geometric mean of \( Z_p \) and \( Z_g \). Such a device is known as the Quarter-Wave Transformer. It should be noted that the device itself is not correctly terminated so that a standing wave will exist along it. The section will always be much shorter, however, than the feeder, and the occurrence of a standing wave within this limited region is not detrimental if the appropriate precautions are taken to avoid spark-over.

Since the characteristic impedances of high-frequency feeders are very nearly purely resistive, it follows from the above equation that the quarter-wave transformer by itself can provide a match only when the load is also purely resistive. It usually happens that the load contains a reactive component, as illustrated in Fig. 3, where the load is represented as a resistance \( R \) in parallel with a reactance \( X \). In this case it is essential to connect across the load terminals a reactance \( -X \) in order to cancel the load reactance. In practice the added reactance usually consists of a short length of transmission line either open or shorted and having its length adjusted to give the required reactance.*

Before attempting to match by use of a quarter-wave transformer, it is desirable that the terminating impedance

\[
\frac{z_0}{z_{\lambda/4}} = \sqrt{Z_0 R_f}
\]

FIG. 3.—Quarter-wave matching network for coupling a feeder to a reactive load.

should be known so that a matching section of the correct characteristic impedance can be constructed and employed. Under circumstances where the pattern of the standing wave which it is desired to remove can be delineated, the value of this impedance can be determined beforehand. Methods are available, however, which do not require these preliminaries. They involve the use of short lengths of line, known as Stubs, in a manner analogous to the use of the \( Z \) line in Fig. 3. Two such methods are illustrated in Fig. 4.

The stub or stubs are located close to the load, and in the system of Fig. 4 (a) a single stub of variable length and variable position is used. Now at both the voltage

* See p. 95.
nodes and anti-nodes along the unmatched feeder, the impedance looking in the direction of the load is purely resistive, with the difference that at the nodes it is less, and at the anti-nodes greater, than the characteristic impedance of the feeder. At some intermediate point the shunt resistance must equal $Z_0$, but at this point, as mentioned previously, there will also be a shunt reactance. The stub of Fig. 4 (a) is placed at this point and its length adjusted to provide an equal and opposite reactance. The feeder is then effectively terminated in its characteristic impedance at the point of stub attachment.

In the system of Fig. 4 (b) there are two stubs, each of variable length, separated by a fixed distance $l$. This length, $l$, of line acts as a transformer in a manner similar to the quarter-wave transformer. The principle of the matching process is as follows. The stub $A$ nearest to the load is adjusted until the shunt resistance component of the line impedance at $B$ is equal to the characteristic impedance. The stub $B$ is then adjusted to cancel out the shunt reactance at $B$, leaving the feeder to the left of $B$ effectively terminated at this point in the purely resistive impedance $Z_0$.

This arrangement is not capable of matching over the whole range of possible terminating impedances, but where it is applicable it represents probably the simplest mechanically of all the possible matching devices. It can be made universal, however, by the addition of a third stub, as discussed in Chapter VI.

In practice, the process of stub matching finally reduces to one of trial and error, the stubs being adjusted step by step until minimum standing wave is observed on the feeder or until maximum power is obtained in the load. Where it is inconvenient or impracticable to observe the voltage or current distribution, the efficacy of the adjustments can be checked by making a deliberate change in the feeder length. This may be effected by inserting a short extra length of feeder between a suitably placed and suitably designed plug-and-socket connector. When the feeder is correctly terminated the change in length will not affect the conditions at either the generator or the load, provided always that the junctions of the added length do not introduce spurious reflections. This added length should preferably be nearer to an odd than to an even number of quarter wavelengths.

**TRANSMISSION LINES IN VALVE OSCILLATORS**

A further application of the low-loss reactance property of short lengths of transmission lines occurs in valve oscillators where they may be used to tune to resonance the valve electrode capacitances. Several examples of this application are illustrated in Fig. 5.

Fig. 5 (a) shows a split-anode magnetron where the capacitance between the two anodes is tuned to resonance by means of a short-circuited parallel wire line. Fig. 5 (b) and 5 (c) show alternative ways in which lines can be used as the inductive reactance between the anode and the grid of a triode self-oscillator. In Fig. 5 (b) a short-circuited twin line is used with a blocking condenser in one line. The steady voltages are supplied through high-frequency chokes. Fig. 5 (c) is similar to Fig. 5 (b), except that the blocking condenser has been inserted at the shorted end of the line, with the advantage that both anode and grid steady

* See Bibliography on page 149.
voltages can be applied at the low potential end of the high-frequency circuit. An amplifying circuit for use with an 'acorn' pentode is shown diagrammatically in Fig. 5 (e).

The pentode has anode and grid leads at opposite ends and the remaining leads are arranged in a ring seal in the middle of the valve envelope. The anode and the grid leads are connected to the inner conductors of two coaxial lines. The outer conductors and the remaining electrodes are joined either directly or through condensers to a conducting plate which has a hole in it to accommodate the valve. In each of the above cases the line length is less than quarter wavelength, the actual length being given by the relation

$$\frac{1}{\omega C} = Z_0 \tan \frac{2\pi l}{\lambda} \ldots (1.2)$$

where $C$ is the valve capacitance, $\omega$ the angular frequency and $\lambda$ the wavelength of the oscillations, $Z_0$ the characteristic impedance of the line, and $l$ its length.

Instead of a short-circuited line, a length of open-ended line can be used, as shown in Fig. 5 (d). The steady supplies for the anode and the grid are in this case connected at the voltage node or as near to it as possible. The line length for this circuit for a given wavelength exceeds that given by equation (1.2) by a quarter of a wavelength. The circuit is therefore useful at the shorter wavelengths when the circuit lengths become small. It has an additional advantage of not requiring a blocking condenser for the d.c. voltage between anode and grid. This can be an important factor in a high-power oscillator.

The limitation imposed on the wavelength by the length of the external circuit has already been mentioned. From equation (1.2) it can be seen that the circuit length $l$ can be increased for a given wavelength by reduction of $Z_0$. This has led to the introduction of special valves which have

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**Fig. 5.**

(a) Split anode magnetron oscillator.
(b), (c) and (d) Triode oscillators.
(e) Pentode amplifier.

**Fig. 6.**—Triode oscillator with the anode and grid forming parts of a concentric line oscillatory circuit.
their electrode leads in the form of coaxial conductors of low characteristic impedance. Fig. 6 shows a circuit for such a valve. The anode and grid electrodes are connected to the two conductors of a coaxial line; indeed, they may be considered as integral parts of the line. Since none of the high-frequency field extends outside the outer conductor of a coaxial line, that conductor may be considered as being at earth potential. The anode being part of this conductor is therefore at earth potential. The cathode of the valve is connected to the inner conductor of another coaxial line, the outer of which is also at earth, i.e. at anode, potential. Thus effectively the grid and the cathode have coaxial line circuits between them and the anode. It will be seen that all the oscillating conductors are completely enclosed in this circuit. This is of considerable importance at short wavelengths since it reduces the possibility of stray couplings and radiation losses.

An alternative to the coaxial line has been used in the double-ended 'door knob' type of valve. In these valves, leads to the anode and to the grid are taken out at both ends of the valve and an open-ended twin line is connected to each pair of leads, as shown in Fig. 7. Both lines are surrounded by screening tubes. The directly heated filament has both leads tuned by means of short lengths of short-circuited coaxial lines.

Fig. 7—Double-ended triode oscillator.

Where the lines employed in the oscillators referred to above are short-circuited they will usually be less than \( \frac{\lambda}{4} \) in length, and where open-ended, between \( \frac{\lambda}{4} \) and \( \frac{\lambda}{2} \), in length. It will be noted, however, that (1.2) is satisfied by an infinite number of values of \( l \) differing successively by half a wavelength. Thus as the wavelength is reduced and the shorting bridge becomes too close to the valve envelope, satisfactory operation can often be obtained by withdrawing the shorting bridge a distance of half a wavelength or a wavelength, or even more. By this means it is possible to operate valves at wavelengths at which the first voltage node is well inside the valve envelope. Even when the node is still outside the valve an extra half-wavelength is sometimes used to facilitate coupling between the oscillator and its load.

LINE RESONATORS AS FREQUENCY STABILIZERS *

Lines of length one quarter-wave which are open at one end and shorted at the other provide resonant circuits of very high Q value at short wavelengths, and may be used to stabilize the frequency of self-oscillating valve circuits in much the same manner as quartz crystals are used at lower frequencies.

Fig. 8(a) illustrates the use of a \( \frac{\lambda}{4} \), length of coaxial line and Fig. 8(b) of a \( \frac{\lambda}{4} \), length of twin line in this way. In the latter case it is essential to enclose the line inside a screen to prevent radiation losses and a resulting reduction in the Q value. In resonators to be employed for stabilizing purposes every possible precaution should be taken to reduce losses to a minimum, such as the use of silver-plated conductors of large diameter; to avoid insulating supports as far as practicable, particularly at the open end.

* See Bibliography on page 149. † See p. 99.
and to ensure good electrical connexion at the short-circuited end of the resonator. In the case of a silver-plated coaxial resonator the optimum ratio of the diameters of the two conductors is about 3:6:1.

**MISCELLANEOUS USES OF LINE CIRCUITS**

A further important sphere of application of transmission lines at very high frequencies is that of measurement. Thus they are used as wavemeters, for the measurement of impedance,* for determination of the permittivity and power factor of dielectric materials † and as component parts of the devices employed for the measurement of current, voltage, and power.

In addition, they find application as:

(a) Inductive and capacitive circuit elements in filter networks.‡

(b) Metallic insulators for supporting circuits carrying V.H.F. currents of constant frequency.

This application arises from the property that the input impedance of the short-circuited line is a pure resistance of very high value.§

(c) A non-contact short circuit.

Since the input impedance of the open-ended \( \frac{\lambda}{4} \) line is a very low resistance, such a line acts sensibly as a short circuit to currents of wavelength \( \lambda \), but as an open circuit to direct and low-frequency alternating currents.*

(d) A condenser which will pass direct currents.

This application results from the fact that the input impedance of short-circuited lines of length between \( \frac{\lambda}{4} \) and \( \frac{\lambda}{2} \) is capacitive.†

* See p. 94. † See p. 95.

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Fig. 8.—Self-oscillators stabilized by means of high-Q resonators

The anode circuits may be line circuits if desired.
CHAPTER II

THE BASIC EQUATIONS FOR TRANSMISSION LINES PROPAGATING IN THE PRINCIPAL MODE

In introducing the subject matter of his recently published monograph on waveguides Dr. Lamont* pointed out that electromagnetic waves may be regarded as divisible into two main types, free waves and guided waves. The radiation from a dipole situated in free space is an example of the first type, while the propagation of energy along transmission lines, consisting either of two coaxial conductors, Fig. 9 (a), or of two parallel wires without or with a screen, Fig. 9 (b) and (c), or along waveguides—that is, through the interior of a single hollow conductor—are examples of the second type. In so far, however, as the only physical distinction between these two forms of propagation is that the electric and magnetic field components of free waves spread more or less in all directions in space, whereas in the case of guided waves they are confined to the vicinity of the guiding system, the approach to general mathematical solutions is in all cases analogous. It involves initially formulation of Maxwell's field equations for the electric field intensity $E$ and magnetic field intensity $H$ in the dielectric medium through which the energy propagation occurs, and finally substitution of the boundary conditions appropriate to the particular system under investigation.

MAXWELL'S EQUATIONS FOR A NON-COCONDUCTING DIELECTRIC MEDIUM

The general form of Maxwell's equations for a homogeneous isotropic non-conducting medium of permittivity

* Methuen monograph, Wave Guides.
\( \varepsilon \) and permeability \( \mu \), in which the electric and magnetic fields \( E \) and \( H \) are continuous, is

\[
\text{curl } H = \frac{\varepsilon}{\varepsilon_0} \frac{\partial E}{\partial t} \ldots \ldots (2.1)
\]

\[
\text{curl } E = \mu \frac{\partial H}{\partial t} \ldots \ldots (2.2)
\]

\[
\text{div } H = \text{div } E = 0 \ldots \ldots (2.3)
\]

In these equations \( \text{curl } H \) is a vector having rectangular components \( \frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}, \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} \); \( \text{curl } E \) is a vector of like form, and \( \text{div } H \) and \( \text{div } E \) are respectively

\[
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \quad \text{and} \quad \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}.
\]

These equations hold at any point where the fields are changing continuously. At a surface of discontinuity, however, certain boundary conditions, which may be derived from the above equations by assuming that the discontinuity is the limiting case of a more and more rapid continuous change, have to be satisfied. They are:

1. the tangential component of \( E \) must be continuous across the surface;
2. the normal component of \( \mu H \) must be continuous across the surface;
3. the discontinuity of the normal component of \( \varepsilon E \) is equal to the surface charge density;
4. the discontinuity of the tangential component of \( H \) is equal to the surface current density.

The particular solution of Maxwell’s equations which satisfies the boundary conditions imposed by the physical

* In this representation of Maxwell’s equations \( E, H, \varepsilon \) and \( \mu \) conform to the rationalized M.K.S. system of units in which \( E \) is expressed in volts per metre, \( H \) in amperes per metre, \( \varepsilon \) in farads per metre and \( \mu \) in henrys per metre. For free space \( \varepsilon \) has the value \( \frac{1}{4\pi} \times 10^{-9} \) farads per metre and \( \mu \) the value \( 4\pi \times 10^{-7} \) henrys per metre.

nature of the system under investigation determines the possible modes of electromagnetic wave propagation in that system. The solutions worked out for waveguides of rectangular, circular, and other sections, for example, are reproduced in detail in Lamont’s book. It is shown that there exists theoretically a succession of possible modes of wave propagation, each mode having characteristic electric and magnetic field configurations. These modes fall into two classes—one, known as \( E \) or transverse magnetic waves, with which there is a component of electric field, but no component of magnetic field, in the direction of energy propagation; the other, known as \( H \) or transverse electric waves, in which the reverse applies. In practical waveguides, however, the possible forms of wave are restricted by the dimensions of the waveguide section taken in relation to the impressed frequency. Thus for each wave type there is a critical wavelength, fixed by the sectional dimensions, below which its propagation along the guide becomes impossible.

Analogously it may be shown that \( E \) and \( H \) waves can be set up in the coaxial transmission line.* There is, however, an important difference between the two systems, namely, that whereas it is not possible for a wave devoid of both \( E \) and \( H \) components in the direction of propagation to exist in the waveguide,† such a wave is possible in the transmission line. Indeed, this is the mode of propagation to which the contents of this book relate. Since the electric and magnetic fields associated with it are purely transverse it may be distinguished from the \( E \) and \( H \) waves by the name Transverse Electromagnetic, T.E.M. wave.

It is widely spoken of as the Principal Wave. Its critical or cut-off frequency is infinite, so that, as in free space, there is no limit, high or low, to the frequency which can be propagated in this mode. The velocity of propagation is the same as in free space, namely, \( \frac{1}{\sqrt{\mu\varepsilon}} \)

† Lamont, loc. cit., p. 29.
HIGH FREQUENCY TRANSMISSION LINES

(= \(3 \times 10^8\) metres/sec. in air). The wavelength in the line for a given frequency is likewise the free-space wavelength for identical dielectric media in the two cases.

The field configurations of the Principal mode in the coaxial line are shown in Fig. 9 (a), and in the unscreened twin line in Fig. 9 (b).

Although it is possible for both \(E\) and \(H\) waves to be propagated in the coaxial transmission line, these modes are not normally utilized. Their excitation, where this occurs, is unintentional and incidental and the dimensions of the coaxial line in use will generally be such as to cause their rapid attenuation, even at the highest frequencies. Thus the critical wavelength of the lowest \(H\) mode—that is, the longest wavelength for which unattenuated propagation of an \(H\) wave is physically possible—is given in terms of the radii of the inner and outer conductors, \(a\) and \(b\) respectively, by

\[\lambda_c \approx \pi(a + b)\]

the mean circumference of the inner and of the inside of the outer conductors, approximately.

The critical wavelength of the lowest \(E\) mode is less than the above; it is

\[\lambda_e = 2.61b \quad \text{for } a \ll b\]

\[\approx 2(b - a) \quad \text{for } \frac{a}{b} > 0.5\]

Thus for a coaxial line of \(a = 0.2\) cm., \(b = 1.0\) cm., no \(H\) or \(E\) mode can be freely propagated for exciting wavelengths longer than about 4 cms.

The longer the exciting wavelength \(\lambda\) relative to the critical or cut-off wavelength \(\lambda_c\), for a particular mode, the more rapidly will the amplitude of the mode be attenuated with distance from its source. If we define the range \(r\) of a mode as the distance in which its amplitude falls to \(\frac{1}{e}\) of its initial value, it may be shown that

\[r = \frac{\lambda}{2\pi \sqrt{\left(\frac{\lambda}{\lambda_c}\right)^2 - 1}}\]

For the lowest \(H\) mode this becomes

\[r = \frac{\lambda}{2\pi \sqrt{\left(\frac{\lambda}{\lambda_c}\right)^2 - 1}}\]

and since this mode has the highest cut-off wavelength (lowest cut-off frequency) this expression gives an upper limit to the range of non-propagated modes. If the exciting wavelength is large compared with this longest critical wavelength, the expression for \(r\) reduces to

\[r \approx \frac{a + b}{2}\]

EFFECT OF DIELECTRIC CONDUCTION AND CONDUCTOR RESISTANCE

When the dielectric medium has a finite conductivity \(\sigma\) the form of equation (2.1) is modified by the addition to the dielectric displacement current \(\varepsilon \frac{\partial E}{\partial t}\) of a conduction current term \(\sigma E\). Thus this equation becomes

\[\sigma E + \varepsilon \frac{\partial E}{\partial t} = \text{curl} \, H \quad \ldots \quad (2.4)\]

The effect of this addition on the solution for particular cases, as compared with that for the loss-free dielectric, is the association with the expressions for the various field components of an attenuation factor involving \(\sigma\). Thus the energy dissipation associated with the conductivity results in an exponential decay of the field components of the propagated wave with distance from the source, but in no way alters the field configuration at any given point in

* Expressed in reciprocal ohm-metre units.
the medium from that which would exist in the absence of conductivity.

There is a further source of energy loss in waveguide and transmission-line systems, however, which cannot be taken account of so easily, namely, the loss in the resistance of the guiding conductors. The existence of conductor resistance involves inevitably a difference in field configuration compared with that for resistanceless conductors, since in the former case current flow in the conductors necessarily requires a longitudinal component of electric field at the metallic surfaces. Thus it is correct to speak of a purely transverse wave in the transmission line only if the guiding conductors are of zero resistance. This ideal condition is assumed by Lamont in his derivation of the permissible field patterns in waveguide systems. Provided, however, that the resistance of the walls of the waveguide is small, these idealized field patterns are substantially correct and the effect of the wall resistance on the wave attenuation may justifiably be calculated in terms of them. Fortunately, the complexity associated with this delayed consideration of the effect of conductor resistance on the propagation can be avoided in the case of the transmission line. Along with the dielectric conductivity it may be taken account of directly in the basic equations in the manner now to be discussed.

THE BASIC EQUATIONS OF THE TRANSMISSION LINE*

(a) Conductors of Zero Resistance. Consider the elements of twin and coaxial lines shown in Fig. 10 in which $AB$ and $CD$ represent reference planes crossing the lines perpendicularly at unit (small) distance apart. It will be assumed initially that the conductors are devoid of resistance so that no reservation attaches to speaking of a purely transverse wave of which the electric and magnetic field components at $AB$ and $CD$ lie entirely in these $yz$, planes. Their configurations in these planes are shown in Fig. 9 (b) and (a) for the two forms of line, the electric and magnetic fields being at right angles at all points. The particular form of equations (2.2) and (2.4) for these systems will now

![Fig. 10.](image)

be derived by considering any pair of corresponding points $O$ and $P$ in the planes $AB$ and $CD$ respectively, Fig. 11, at which the electric and magnetic fields have magnitude $E, H$ and $E + \frac{\partial E}{\partial x} \times 1, H + \frac{\partial H}{\partial x} \times 1$, directed along arbitrary $y$ and $z$ axes.

![Fig. 11.](image)

First consider the elementary rectangle $OSTP$. The total flux crossing the surface area of this rectangle is $\mu H \times 1 \times \delta y$ and by Faraday's law of induced E.M.F. the

rate of change of this flux is related to the line integral of \( E \) round the rectangle in the manner
\[
\left( E + \frac{\partial E}{\partial x} \times 1 - E \right) \delta y = -\mu \frac{\partial H}{\partial t} \times 1 \times \delta y
\]
\[
\frac{\partial E}{\partial x} = -\frac{\partial H}{\partial t}
\]
(2.5)
to the first order of small quantities. This corresponds to equation (2.2).

Now consider the elementary rectangle \( OPQR \). The total current crossing this surface due to the electric field strength \( E \) is \( \left( \sigma E + \varepsilon \frac{\partial E}{\partial t} \right) \times 1 \times \delta z \). By the work law, and employing the units given on page 18, the line integral of \( H \) around this rectangle is equal to this enclosed current. Thus
\[
\left( H - H - \frac{\partial H}{\partial x} \times 1 \right) \delta z = \left( \sigma E + \varepsilon \frac{\partial E}{\partial t} \right) \times 1 \times \delta z
\]
\[
\frac{\partial H}{\partial x} = -\left( \sigma E + \varepsilon \frac{\partial E}{\partial t} \right)
\]
(2.6)
corresponding to equation (2.4).

The dependent variables in these equations (2.5) and (2.6) are the electric and magnetic field intensities \( E \) and \( H \). In order that the effect of conductor resistance may be included conveniently it is necessary, however, to replace these variables by their line integrals, namely, the voltage between and the current flowing in the conductors, respectively. This is not to say that it is readily—if at all—possible to measure either of these quantities directly in line systems operating at very high frequencies, but this does not detract from the analytical advantages associated with this transformation. It involves, of course, other necessary transformations.

Consider the plane \( AB \) of Fig. 10 crossing the conductors at right angles. If it be assumed for the moment that these conductors have no resistance, the field components of the wave at this point along the line lie entirely in this plane—there is no tangential component of \( E \) on the surface of the conductors and no component of \( H \) parallel to them. The voltage \( v \) between the conductors at \( AB \) at any instant is the line integral of \( E \) from one conductor to the other taken along any path in this reference plane. Similarly, the line integral of \( H \) round any path in this plane embracing one of the conductors is a measure of the instantaneous current \( i \) in that conductor.

Now consider the element of line between the two planes \( AB \) and \( CD \). At every point in the dielectric between these planes the flux density \( B = \mu H \), the precise values of \( B \) and \( H \) varying, of course, but in constant ratio, from point to point. If, now, \( H \) as it appears in (2.6) is to be replaced by its line integral \( i \), and \( E \) as it appears in (2.5) by its line integral \( v \), so \( \mu H = B \) in the former equation must be replaced by its integral taken over the whole surface between the reference planes. This means the total magnetic flux passing between and round the conductors, between these planes. But by the definition of inductance this may be expressed as \( LI \), where \( L \) is the inductance of the unit length of line.

Hence equation (2.5) may be replaced by
\[
\frac{\partial v}{\partial x} = -L \frac{\partial i}{\partial t}
\]
(2.7)
an equation which is normally derived by applying the technique of electric circuit analysis to the elementary circuit bounded by the conductors and the reference planes \( AB \), \( CD \). The voltage between the conductors at \( AB \) is \( v \) and at \( CD \) \( v + \frac{\partial v}{\partial x} \times 1 \), and since there is no longitudinal component of electric field the summation of voltage around the circuit is \( \frac{\partial v}{\partial x} \times 1 \), a quantity which is also given by \( -L \frac{\partial i}{\partial t} \), where \( L \) is the inductance of the circuit, that is, the inductance per unit length of line.
The transformation of equation (2.6) follows in a similar manner. Thus the integral of the purely transverse conduction current density $\sigma E$ taken between the reference planes is $Gv$, where $G$ is the transverse conductance of the dielectric medium per unit length of line, while the corresponding integral of the displacement current density $\varepsilon \frac{\partial v}{\partial t}$ is $C\frac{\partial v}{\partial t}$, where $C$ is the capacitance per unit length. Equation (2.6) may therefore be replaced by

$$\frac{\partial i}{\partial x} = -\left(Gv + C\frac{\partial v}{\partial t}\right) \ldots \ldots (2.8)$$

which states that the current $i$ in the conductor elements between the reference planes decreases by an amount $\frac{\partial i}{\partial x} \times 1$ equal to the conduction current $Gv$ and the time rate of change of the charge $Cv$ on these elements.

(b) Conductor of Finite Resistance. The existence of conductor resistance demands a component of electric field tangential to the conductor surfaces; the lines of electric force no longer leave the positive, and enter the negative, conductor at right angles so that the wave is no longer a purely transverse one. If, however, the resistance is small the departure from planarity will be small, though its effect on the wave propagation is nevertheless of the greatest importance.

At this point the author cannot do better than quote direct from Heaviside.*

We may still regard $v$ as the transverse voltage in the reference plane. Whether we go straight across from the positive to the negative lead, keeping in the reference plane, or leave it slightly in order to precisely follow a line of force in estimating the voltage, is of no moment, because the results differ so slightly. The quantity $C$ [the $i$ in the preceding] is subject to a similar slight difference, according to the way it is reckoned. We may perhaps most conveniently


reckon it as the circulation of the magnetic force round either lead upon its surface and in the reference plane, because this way gives the exact value of the conduction current in the lead. But the circulation in other paths in the reference plane will not give precisely the same value now that the waves are not quite plane. This difference we also ignore in the practical theory. In truly plane waves the electric current in the dielectric is wholly transverse. There is now really an axial component as well, but being a minute fraction of the transverse current, it is ignored. In short, we have to make believe that the waves are planar when considering their propagation through the dielectric, while at the same time we take into account the departure from planarity in considering the effect of the leads and what occurs in them. It is unfortunate to have to refer to small corrections, as it confuses the statement of the vitally important matters. Let us, then, set them aside now.

Suppose now that the tangential component of electric field at the surface of the positive conductor is $E_1$ and that at the surface of the negative one $E_3$, but of opposite sign, and again add up the voltages round the elementary circuit bounded by the conductors and the reference planes $AB, CD$. The summation is

$$E_1 + \left(v + \frac{\partial v}{\partial x}\right) + E_3 - v = \frac{\partial v}{\partial x} + E_1 + E_3$$

so that equation (2.7) now takes the form

$$\frac{\partial v}{\partial x} = -\left(E_1 + E_2 + L\frac{\partial i}{\partial t}\right) \ldots \ldots (2.9)$$

Equation (2.8) is unchanged since, as indicated in the preceding paragraph, the minor changes as regards $v$ and $i$ are not significant.

Again quoting from Heaviside, with minor modifications:

* loc. cit., p. 388.
continue to be \( v \) and \( i \), we require to express \( E_1 \) and \( E_2 \) as functions of \( i \). Fortunately this can be done, sometimes very simply, and at other times in a more complicated way. To take the simplest case let the leads be tubes (or sheets) of so small depth that penetration is practically instantaneous as waves pass along them. Then \( E_1 \) is not only the boundary tangential electric force, but is also the axial electric force throughout the substance of the positive tube. Similarly as regards \( E_2 \) for the negative tube. It is true that, in virtue of the transverse electric current from tube to tube, there is also transverse current in the tubes, and, therefore, transverse electric force, but this is to be ignored because it is a small fraction of the axial. The current density in the positive tube is therefore axial and of strength \( g_1E_1 \) where \( g_1 \) is the conductivity of the material, and the total current \( G_1E_1 \) where \( G_1 \) is the axial conductance per unit length or, which is the same, \( \frac{E_1}{R_1} \), where \( R_1 \) is the resistance per unit length. But this is also the quantity \( i \), so that we have the elementary relations

\[
E_1 = R_1 i; \quad E_2 = R_2 i
\]

which are essentially the connexions of the electric and magnetic forces at the boundaries, though brought to a particularly simple form by the instantaneous penetration.

Equation (2.9) thus takes the form

\[
\frac{\partial v}{\partial x} = -\left(Ri + \frac{\partial h}{\partial t}\right)
\]

where \( R = R_1 + R_2 \) is the combined resistance per unit length of the two conductors.

In a succeeding section of his work Heaviside discusses the more general case in which the penetration is not instantaneous and shows that where the voltage and current vary periodically with time at frequency \( f \), equation (2.10) is still valid provided that the resistance \( R \) to steady current is replaced by the effective resistance of the conductors at this frequency and \( L \) by the corresponding effective inductance. Expressions for these quantities are derived in the next chapter, and are implicit in all the succeeding calculations.

### THE EFFECT OF LINE DISCONTINUITIES

Before passing on to the solution of the basic line equations (2.8) and (2.10) it must be emphasized that these equations relate to undisturbed propagation of the Principal Wave along a uniform line. Later, however, we shall be concerned with terminating finite lengths of line and with discontinuities in the geometrical form of the line conductors or in the dielectric medium. In the vicinity of these discontinuities the electromagnetic field distribution will, in general, differ from the simple one which characterizes the principal wave and is no longer representable in terms of the above-mentioned equations. The complete solution of each case requires in fact an independent solution of Maxwell's equations which satisfies all the necessary boundary conditions, and its derivation may involve great mathematical difficulties. The complex field distribution may be represented as a superposition of the fields of the energizing principal wave and of higher modes such as were discussed on pages 19-21, and from what was said there it will be evident that the field complexity will extend further and further down the line from the discontinuity the closer the principal wavelength approaches the critical wavelength of the lowest of these modes. In the usual low-frequency applications of line theory it is justifiable to ignore these effects completely, but this is not so at very high frequencies, particularly where the length of line involved is only a fraction of the principal wavelength.

This matter is dealt with more fully in Chapters IV and V.
CHAPTER III

THE PROPAGATION CHARACTERISTICS OF LINES

GENERAL SOLUTION OF THE LINE EQUATIONS

It has been shown that subject to an approximation concerning the conductor resistance, the field equations for the uniform line propagating the principal mode may be transformed into the equations

\[
\frac{dI}{dx} = -(Gv + C\frac{dv}{dt})
\]

\[
\frac{dv}{dx} = -(Ri + L\frac{di}{dt})
\]

in which \( v \) and \( i \) are the instantaneous values of the line voltage and current at an arbitrary point \( x \), and \( C, L, R \) and \( G \) are 'constants' of the line per unit length.

The particular solution of these equations in which we are interested is that for which \( v \) and \( i \) are simple harmonic functions of the time \( t \). Thus suppose them to be given by \( V e^{j\omega t} \) and \( I e^{j\omega t} \) respectively, \( V \) and \( I \) being the amplitudes of the time variable voltage and current at the point \( x \). On making these substitutions for \( v \) and \( i \) the above equations become

\[
\frac{dV}{dx} = -(R + j\omega L)I
\]

\[
\frac{dI}{dx} = -(G + j\omega C)V
\]

in which \( e^{j\omega t} \) is a common factor throughout.

The solution of these equations gives the voltage and current time amplitudes—quantities which will, generally speaking, have both real and imaginary parts—as functions of the distance \( x \) along the line from a selected reference point.

On eliminating \( I \)

\[
\frac{d^2V}{dx^2} = (R + j\omega L)(G + j\omega C)V = P^2V \text{ say}
\]

where

\[
P = \sqrt{(R + j\omega L)(G + j\omega C)} \quad \ldots \quad (3.2)
\]

The general solution of this equation is

\[
V = Ae^{-px} + Be^{px} \quad \ldots \quad (3.3)
\]

in which \( A \) and \( B \) are arbitrary constants.

But from (3.1)

\[
I = -\frac{1}{R + j\omega L} \frac{dV}{dx} = \frac{P}{(R + j\omega L)}(Ae^{-px} - Be^{px})
\]

\[
= \left(\frac{G + j\omega C}{R + j\omega L}\right) \frac{I}{(R + j\omega L)}(Ae^{-px} - Be^{px})
\]

\[
= \frac{1}{Z_0}(Ae^{-px} - Be^{px}) \quad \ldots \quad (3.4)
\]

in which

\[
Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \ldots \quad (3.5)
\]

The values to be attached to the arbitrary constants \( A \) and \( B \) result when \( V \) and \( I \) are specified at some suitable reference point. This we shall delay doing until the next chapter. Meanwhile, we shall examine the physical significance of equations (3.3) and (3.4) and, for this purpose, express the complex quantity \( P \) in the form \( \alpha + j\beta \). Both \( \alpha \) and \( \beta \) will prove to be positive quantities. Equation (3.3) now becomes

\[
V = Ae^{-\alpha x} \cdot e^{-j\beta x} + Be^{\alpha x} \cdot e^{j\beta x}
\]

The first term represents the voltage component of a wave which decreases exponentially with \( x \) at a rate \( e^{-\alpha} \) per unit length of line in the positive \( x \) direction, and which in the same distance changes in phase by an angle \(-\beta\)
rad
ts. This corresponds to a wave travelling along the line in the positive \( \alpha \) direction, that is away from the source at \( x = 0 \), one wavelength being traversed for each \( 2\pi \) change of \( \beta x \).

The second term represents the voltage component of an exactly similar wave travelling in the reverse direction, namely, towards the source. We shall have to discuss later the physical mechanism by which this backward wave is generated.

The expression for the current \( I \) is made up similarly, but with the important difference that the phase of the backward-travelling current component is reversed relative to that of the corresponding voltage component.

Since \( \alpha \) determines the rate of amplitude change, and \( \beta \) the rate of phase change, with distance \( x \), they are named the \textit{Attenuation} and \textit{Phase} constants, respectively, their resultant \( P \) being called the \textit{Propagation Constant}.

If, now, the line is regarded as of infinite length, and we make the substitution \( V = I = 0 \) at \( x = \text{infinity} \) in (3.3) and (3.4), the backward wave components must be zero for all values of \( x \), and in consequence

\[
V = Ae^{-\alpha x}e^{-j\beta x} = V_s e^{-\alpha x} e^{-j\beta x} \quad \text{and} \quad I = \frac{V_s}{Z_0}e^{-\alpha x} e^{-j\beta x},
\]

where \( V_s \) is the input or sending end voltage at \( x = 0 \), and \( \frac{V}{I} = Z_0 \) throughout the line length. This means that the effective impedance of the infinite line has the value \( Z_0 \) at every point; \( Z_0 \) is thus a characteristic property of the line and it is called the \textit{Characteristic Impedance}. The infinitely long line is clearly a purely hypothetical conception, but the result associated with it is of the greatest importance since, as we shall see, it represents the behaviour of a finite length of line when this is terminated in such a manner that there is no resultant reflected wave. Such a line is known as a \textit{correctly terminated line}.

* See, pp. 59-62.

Before discussing \( \alpha, \beta \) and \( Z_0 \) in detail we must consider the factors governing the quantities \( C, L, R \) and \( G \) on which they depend.

\textbf{CONDUCTOR AND DIELECTRIC ASSEMBLIES EMPLOYED AT HIGH FREQUENCIES}

Three forms of conductor arrangement find application as high-frequency transmission lines, namely, the 'balled' open-wire twin and the screened twin forms, and the 'unbalanced' coaxial or concentric (Fig. 9). The nature of the conductors and of the dielectric separation between them varies considerably, however, depending on the intended application. Thus where short lengths of line are to be used as circuit elements the conductors will consist in general of cylindrical copper, brass or silver-plated rods or tubes, though it may be convenient to construct the tank circuit of an oscillator, for example, from parallel rods of rectangular section. In these cases the dielectric between the conductors will invariably be air, excepting, of course, for such small pieces of solid dielectric material as are necessary to impart mechanical rigidity to the assembly. Where, however, the line is to serve as an inter-connector or feeder between well-separated pieces of equipment, such as an oscillator and an antenna, and a certain degree of flexibility is required in the interconnexion, the centre conductors may be of stranded copper, and the outer of the coaxial and the screen of the twin either lead, tinned copper wire braid or lapped copper tape. In these cases the dielectric may consist of solid material throughout, the restrictions being that this material must possess adequate flexibility, and sufficiently low dielectric loss at the frequency in question, to meet the mechanical and electrical requirements, respectively. A material which meets these requirements extremely well and is used almost exclusively for solid dielectric lines in this country is the thermoplastic Polythene. * Alternatively, the dielectric space between the

conductors may be occupied mainly by air with the necessary mechanical support provided either by solid dielectric spacers located at intervals along the line, or in the case of the coaxial by a 'thread' of solid material wrapped in an open spiral around the centre conductor. The material usually employed in these cases is the thermoplastic Polystyrene.

It will not be possible within the space of this small book to derive expressions for the capacitance, inductance, resistance and conductance per unit length of the various line assemblies mentioned above. They will be developed for the simplest case only, that of the coaxial line with continuous solid dielectric separation between smooth conductors, and even then only in respect of the resistance and conductance. Derivation of the expressions for the capacitance and inductance of this line is considered unnecessary since the low-frequency expressions are applicable at very high frequencies, except that in consequence of the minute depth of current penetration into the conductors at the latter frequencies the 'internal' component of inductance ceases to be significant.

**CALCULATION OF CONDUCTOR RESISTANCE AT HIGH FREQUENCY—THE COAXIAL LINE**

Calculation of the high-frequency resistance of a conductor requires a knowledge of the current distribution over the cross-section. The complete solution for isolated conductors of circular section involves Bessel functions, but when the conductor dimensions and the frequency are such that the depth of current penetration is very small compared with the total thickness of metal involved—the conductor radius for a solid conductor and the wall thickness for a hollow one—the investigation may be simplified considerably by treating the conductor surface as part of the surface of a flat conducting slab. This simplification is normally permissible for frequencies in excess of 10 Mc/sec.

The surface of the slab to be considered is represented by the $xy$ plane in Fig. 12; the depth in the $z$ direction is supposed infinite. For current flow in the $x$ direction under a suitably applied electric field, the magnetic field will be directed along the $y$ axis.

Reference to (2.5) and (2.6) will show that the variations with $z$ of the current density $j = \sigma E$ and magnetic field strength $H$ are governed by the equations

\[
\frac{1}{\sigma} \frac{\partial j}{\partial z} = -\mu \frac{\partial H}{\partial t}
\]

\[
\frac{\partial H}{\partial z}
\]

---

the displacement current term in (2.6) being negligible in this case compared with the conduction current one.

On eliminating \( H \) from these equations

\[
\frac{\partial^2 j}{\partial x^2} = \mu \sigma \frac{\partial j}{\partial z}
\]

of which

\[
j = j_s e^{-mz} \cos (\omega t - mz) \quad \ldots \quad (3.6)
\]

will be seen to be a solution. Here \( j_s \) is the amplitude of the current density at the surface of the slab, \( z = 0 \); \( \omega \) is the angular frequency, \( 2\pi f \), of the applied electric field and

\[
m = \sqrt{\frac{\omega \mu \sigma}{2}} = \sqrt{\pi f \mu \sigma} \quad \ldots \quad (3.7)
\]

The magnetic field strength variation with \( z \) and \( f \) is given by (3.6) on replacing \( j_s \) by \( H_s \). Thus the amplitudes of \( j \) and \( H \) decay exponentially with \( z \) and at depth \( 4/m \) become less than 2 per cent of their surface values.

Before proceeding we must gain an idea of the numerical value of \( m \) for copper and lead at the frequencies with which we shall be concerned, say \( 10^8 \) c.p.s. upwards. In C.G.S. E.M. units the conductivities of copper, tin, and lead at room temperature are approximately \( 5.9 \times 10^{-4} \), \( 7.65 \times 10^{-5} \), and \( 4.65 \times 10^{-5} \) per centimetre cube respectively. In the rationalized M.K.S. units they are \( 5.9 \times 10^7 \), \( 7.65 \times 10^6 \), and \( 4.65 \times 10^6 \) mhos per metre cube. Both copper and lead being non-magnetic materials, the permeability of each is that of free space, \( \mu_0 \), which, in the same units, is \( 4\pi \times 10^{-7} \). Thus the value of \( m \) for copper is \( 15.3\sqrt{f} \) and for lead \( 4.3\sqrt{f} \). Accordingly, the current density will have fallen to about 2 per cent of its surface value at depths \( 0.26/\sqrt{f} \) and \( 0.93/\sqrt{f} \) metres respectively. At \( 10^8 \) c.p.s., for example, these depths are \( 0.26 \times 10^{-4} \) and \( 0.93 \times 10^{-4} \) metres (0.026 and 0.093 mm. respectively), values sufficiently small to justify the assumption referred to above for conductors of normal dimensions.

The effective resistance of the slab to current flow of the form given by (3.6) is derived from calculation of the total current and the total mean power loss per unit width of slab.

The former is given by

\[
i = \int_{z=0}^{z=\infty} j \times \text{d}z = j_s \int_{0}^{\infty} e^{-mz} \cos (\omega t - mz) \text{d}z
\]

\[
= \frac{j_s}{\sqrt{2 \pi m}} \cos \left( \frac{\omega t - \pi}{4} \right) = I \cos \left( \frac{\omega t - \pi}{4} \right) \quad \ldots \quad (3.8)
\]

where \( I = \frac{j_s}{\sqrt{2 \pi m}} \) is the amplitude of the total current.

The instantaneous power loss per unit width and length of slab in an element of depth \( dz \) distant \( z \) below the surface is

\[
dW = \frac{j_s^2}{\sigma} e^{-2mz} \cos^2 (\omega t - mz) \text{d}z
\]

\[
= \frac{j_s^2}{2\sigma} e^{-2mz} \{1 + \cos 2(\omega t - mz)\} \text{d}z
\]

The mean value of this with respect to time is thus

\[
j_s^2 \int_{z=0}^{z=\infty} e^{-2mz} \text{d}z
\]

Hence the total mean power loss per unit width and length for the whole depth of slab is

\[
W = \frac{j_s^2}{2\sigma} \int_{0}^{\infty} e^{-2mz} \text{d}z = \frac{j_s^2}{2\sigma} \frac{m}{4\pi \mu \sigma} = \frac{I^2}{2\sigma^2}
\]

But if the current \( i = I \cos \left( \frac{\omega t - \pi}{4} \right) \) were to flow through a resistance \( R \), the mean power loss in this resistance would be

\[
W = \frac{I^2}{2R}
\]

Hence the slab resistance per unit width and length is
equal to \( \frac{m}{\sigma} \), the same as the resistance of a surface skin of thickness \( \frac{1}{m} \) carrying a uniformly distributed current.

On the previous assumption that the surface curvature of cylindrical conductors may be ignored at the frequencies in question, the resistance of an isolated conductor of radius \( a \) per unit length is thus

\[
R = \frac{1}{2\pi a} \frac{m}{\sigma} = \frac{1}{2\pi a} \sqrt{\pi j \mu \sigma} \]

\( 2\pi a \) being the width of the equivalent flat slab.

Since the current distribution in this conductor will be unaffected by enclosing it within a concentric cylindrical conductor of inner radius \( b \), the total resistance per unit length (metre) of the coaxial line of Fig. 9 (a) is given by

\[
R = \frac{1}{\sqrt{4\pi}} \sqrt{\frac{f}{10^7 \left( \frac{1}{a \sqrt{\sigma_a}} + \frac{1}{b \sqrt{\sigma_b}} \right)}} \text{ ohms/metre}
\]

\( a \) and \( b \) being in metres and \( \sigma_a \) and \( \sigma_b \) in mhos per metre cube. \( \mu_a \) and \( \mu_b \) will have the free-space value \( \mu_0 = 4\pi \times 10^{-7} \), so that \( R \) may be rewritten

\[
R = \sqrt{\frac{f}{10^7 \left( \frac{1}{a \sqrt{\sigma_a}} + \frac{1}{b \sqrt{\sigma_b}} \right)}} \text{ ohms/metre} \quad (3.9)
\]

In practice the conductors may be of tinned copper, and it will be appreciated that when the depth of current penetration exceeds the thickness of the tinning the high-frequency resistance will be less than is obtained by substituting the conductivity of tin in this expression. The actual calculation is complex. Furthermore, the centre conductor may be stranded and the outer may be a tinned copper braid. In these cases the evaluation of the respective conductor resistances is difficult, but as a rough approximation they may be taken as 1·25 and 2·4 times the resistances of smooth conductors of the same diameter.

It is of interest to refer back for a moment to equation (3.8). The negative phase angle \( \frac{\pi}{4} \) of the total current indicates that the slab possesses an internal inductive reactance to the current flow equal to its resistance. From this we deduce that the internal inductance of the conductors of the coaxial line has a value given by \( L \omega = R \), that is

\[
L_{\text{internal}} = \frac{1}{2\pi} \sqrt{\frac{f}{10^7 \left( \frac{1}{a \sqrt{\sigma_a}} + \frac{1}{b \sqrt{\sigma_b}} \right)}} \text{ henrys/metre.}
\]

This contribution to the line inductance decreases as the square root of the frequency, and, as mentioned previously, is negligible at very high frequencies compared with that associated with the dielectric space between the conductors.

**CONDUCTANCE PER UNIT LENGTH OF THE COAXIAL LINE**

For a continuous dielectric material of conductivity \( \sigma_d \) it is easily shown that the conductance per unit length of the coaxial line of Fig. 9 (a) is

\[
G = \frac{2\pi \sigma_d}{\log_e^2 \frac{b}{a}} \text{ mhos/metre}
\]

This representation of the conductance is not a convenient one for practical application, however, since the a.c. conductivity of dielectric materials is not normally specified. This imperfection in dielectrics is usually expressed in terms of the tangent of the angle \( \delta \) by which the total current through the medium (the resultant of the displacement and conduction current components) is displaced from exact phase quadrature with the applied voltage. If the capacitance per unit length of line is \( C \) farads, it is evident that

\[
\tan \delta = \frac{G}{C \omega}
\]
High Frequency Transmission Lines

Expressed in the rationalized M.K.S. units, \( C \) for the coaxial line is given by

\[
C = 2\pi \frac{\varepsilon}{b} = \frac{2\pi \varepsilon_0}{\log_e \frac{b}{a}}
\]

where \( \varepsilon_0 = \) permittivity of free space (air) = \( \frac{1}{36\pi \times 10^9} \) farads/metre and \( \frac{\varepsilon}{\varepsilon_0} \) is the relative permittivity of the dielectric medium in question.

Thus \( G = \frac{2\pi}{1.8 \times 10^{10}} \frac{\left(\frac{\varepsilon}{\varepsilon_0}\right) f \tan \delta}{\log_e \frac{b}{a}} \) mhos/metre

For small values of the 'loss angle' \( \delta \),

\[
\tan \delta = \sin \delta = \cos \phi
\]

where \( \phi \) is the phase difference between the voltage and the resultant current, and \( \cos \phi \) the 'power factor' of the material. For good dielectric materials reference to \( \tan \delta \) as the power factor is therefore justifiable.

It will be seen later that the use of materials of power factor as low as 0.0005, that is, of loss angle only 2', can exercise a very serious effect on the performance of transmission lines at very high frequencies. Clearly, then, it is desirable to use air dielectric whenever possible and to keep to a minimum the amount of supporting solid material. Nevertheless, for mechanical reasons, considerable use is made of feeders provided with a continuous solid dielectric core.

\[
\frac{C}{L, R \text{ and } G \text{ expressions for coaxial, unscreened twin and screened twin lines}}
\]

(a) Coaxial Line with Continuous and Composite Dielectrics. At very high frequencies the following expressions give the capacitance, inductance, resistance and conductance of coaxial lines to a degree of accuracy adequate for all practical purposes.

\[
C = 2\pi \frac{\varepsilon}{b} = \frac{1}{1.8 \times 10^{10}} \frac{\left(\frac{\varepsilon}{\varepsilon_0}\right)}{\log_e \frac{b}{a}} \text{ farads/metre}
\]

\[
L = \frac{1}{2\pi \mu_0} \frac{2 \times 10^{-7}}{\log_e \frac{b}{a}} \text{ henrys/metre}
\]

\[
R = \sqrt{\frac{f}{10^9} \left(\frac{1}{a \sqrt{\sigma_a}} + \frac{1}{b \sqrt{\sigma_b}}\right)} \text{ ohms/metre}
\]

where \( a \) and \( b \) are in metres and \( \sigma \) has the value \( 5.9 \times 10^9 \) mhos/metre cube for copper.

\[
G = C \omega \tan \delta = \frac{2\pi}{1.8 \times 10^{10}} \frac{\left(\frac{\varepsilon}{\varepsilon_0}\right) f \tan \delta}{\log_e \frac{b}{a}} \text{ mhos/metre}
\]

The above expressions relate to a continuous dielectric core. Figs. 13 (a) and (b) illustrate elements of coaxial lines in which the dielectric is partly air and partly solid dielectric support, consisting in the first case of disk spacers of width \( D \) and separation \( S \), and in the second of a cylindrical thread of diameter \( (b - a) \) wrapped in the form of a spiral around the centre conductor. In the latter the ratio of the cross-sectional area of the thread to that of...
the total dielectric space is \( \frac{1}{4} \left( \frac{b-a}{b+a} \right) \) and to a fair degree of approximation the thread may be regarded as filling a segment of the dielectric space subtending an angle \( \phi \) at the centre of the line where \( \frac{\phi}{2\pi} \) is equal to the above ratio.

If the relative permittivity and power factor of the solid material are \( \left( \frac{\varepsilon}{\varepsilon_0} \right) \) and \( \tan \delta \)—for the air part these are \( 1 \) and \( 0 \) respectively—and provided, in case (a), that the separation \( S \) of the spacers is a small fraction of the wavelength in the line, the effective values of relative permittivity and power factor for the lines as a whole, for substitution in (3.10), are:

\[
\left( \frac{\varepsilon}{\varepsilon_0} \right)_a = 1 + \left( \frac{\varepsilon}{\varepsilon_0} - 1 \right) \frac{D}{S}; \tan \delta(a) = \frac{\varepsilon \left( \frac{\phi}{2\pi} \right)}{1 + \left( \frac{\varepsilon}{\varepsilon_0} - 1 \right) \frac{D}{S}} \tan \delta
\]

\[
\left( \frac{\varepsilon}{\varepsilon_0} \right)_b = 1 + \left( \frac{\varepsilon}{\varepsilon_0} - 1 \right) \frac{\phi}{2\pi}; \tan \delta(b) = \frac{\varepsilon \left( \frac{\phi}{2\pi} \right)}{1 + \left( \frac{\varepsilon}{\varepsilon_0} - 1 \right) \frac{\phi}{2\pi}} \tan \delta
\]

\[ (3.11) \]

\[
\int \left[ \begin{array}{c}
C = \frac{1}{3.6 \times 10^6} \log \left[ \frac{d}{2a} + \sqrt{\left( \frac{d}{2a} \right)^2 - 1} \right] \text{farads/metre} \\
L = 4 \times 10^{-7} \log \left[ \frac{d}{2a} + \sqrt{\left( \frac{d}{2a} \right)^2 - 1} \right] \text{henrys/metre} \\
R = 2 \sqrt{\frac{f}{10^7 \sigma_a}} \frac{\left( \frac{d}{2a} \right)^2}{a \sqrt{\left( \frac{d}{2a} \right)^2 - 1}} \text{ohms/metre}
\end{array} \right]
\]

A comparison of this expression with that for the resistance of an isolated wire reveals that the 'proximity

---

\‡ E. B. Moullin, Principles of Electromagnetism, p. 245.
HIGH FREQUENCY TRANSMISSION LINES

The effect between the two wires of the twin increases the resistance of each conductor by the factor

$$\frac{d}{2a} \sqrt{\left(\frac{d}{2a}\right)^2 - 1}$$

$$G = C\omega \tan \delta \text{ mhos/metre}$$

as before, but with C now as stated above.

(c) Screened Twin Line with Continuous Dielectric.

Fig. 9 (c).

$$C = \frac{1}{3.6 \times 10^{40}} \left(\frac{\varepsilon}{\varepsilon_0}\right) \log_e \frac{d}{a} \frac{1 - \left(\frac{d}{2s}\right)^2}{1 + \left(\frac{d}{2s}\right)^2} \text{ farads/metre}$$

$$L = 4 \times 10^{-7} \log_e \frac{d}{a} \frac{1 - \left(\frac{d}{2s}\right)^2}{1 + \left(\frac{d}{2s}\right)^2} \text{ henrys/metre}$$

$$R = \frac{2}{a} \sqrt{\frac{f}{10^7\sigma_a}} + \frac{8}{5} \sqrt{\frac{f}{10^7\sigma_a}} \left\{\frac{\left(\frac{d}{2s}\right)^2}{1 - \left(\frac{d}{2s}\right)^4}\right\} \text{ ohms/metre}$$

$$G = C\omega \tan \delta \text{ mhos/metre}$$

The expressions for C, L and R above are approximate forms of complex expressions for these quantities derived in somewhat similar ways by Kirschstein,* Kaden,† and Green, Leibe and Curtis,* but are sufficiently accurate for most practical cases. The first term of the expression for R relates to the centre conductors, and it will be noted that it does not contain a proximity effect factor. This is to be accounted for partly by the approximation, but it is bound up with the fact that the tendency of each centre conductor to disturb the uniformity of charge distribution on the surface of the other is counteracted by the presence of the screen.

It is worth while to obtain some idea of the relative magnitudes of the screen and centre conductor loss resistance terms. Thus for the particular values $\frac{s}{a} = 6$ and $\frac{d}{2s} = 0.5$ and for a lead screen and copper centre conductors, the screen term—the second—is 60 per cent of the centre conductor one; that is,

$$R = \frac{3}{a} \sqrt{\frac{f}{10^7\sigma_a}}.$$

For a coaxial line having lead outer and copper centre conductor of relative dimensions $\frac{b}{a} = 6$, $R = \frac{1.56}{a} \sqrt{\frac{f}{10^7\sigma_a}}$, from which it will be seen, not surprisingly, that the conductor loss in coaxial lines is substantially less than in similarly dimensioned screened twin ones.

THE CHARACTERISTIC IMPEDANCE $Z_0$

According to (3.5) the characteristic impedance of a line propagating in the principal mode is given in terms of the quantities C, L, R, and G by

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Since at very high frequencies R increases as $\sqrt{f}$, whereas $\omega L$ increases directly as $f$, the ratio $\frac{R}{\omega L}$ decreases as $\sqrt{f}$.

* E.N.T., 1936, 13, p. 286.
Let us take a coaxial line of copper conductors with dimensions \( a = 0.05 \text{ cm} = 5 \times 10^{-4} \text{ metres}, \ b = 0.5 \text{ cm} = 5 \times 10^{-3} \text{ metres} \). From (3.10),

\[
R = 9.05 \times 10^{-5} \sqrt{f} \text{ ohms/metre} ;
\]

\[
L_0 = 2.9 \times 10^{-6} f \text{ ohms/metre} ; \quad \frac{R}{L_0} = \frac{31}{\sqrt{f}}.
\]

Thus at

10^8 \text{ c.p.s.} \quad \frac{R}{L_0} = 3.1 \times 10^{-8} \text{, which is negligible compared with unity} ; \text{ it will clearly become still more negligible at higher frequencies.}

It has also been stated that \( \frac{G}{\omega C} = \tan \delta \) for the dielectric space should desirably approximate to zero. Under these circumstances the characteristic impedance is given to a high degree of accuracy by the simple expression

\[
Z_0 = \sqrt{\frac{L}{C}} \text{ ohms}
\]

This expression is entirely real, so that \( Z_0 \) has the character of a pure resistance and the voltage and current wave components in the infinite line are in phase in space throughout the line length.

On substituting for \( C \) and \( L \) from (3.10), (3.13) and (3.14) the following expressions for \( Z_0 \) result

Coaxial line

\[
Z_0 = 60 \sqrt{\frac{\varepsilon_0}{\varepsilon}} \log_{10} \frac{b}{a} = 138 \sqrt{\frac{\varepsilon_0}{\varepsilon}} \log_{10} \frac{b}{a} \text{ ohms}
\]

Unscreened twin

\[
Z_0 = 276 \sqrt{\frac{\varepsilon_0}{\varepsilon}} \log_{10} \left\{ \frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1} \right\} \text{ ohms}
\]

Screened twin

\[
Z_0 = 276 \sqrt{\frac{\varepsilon_0}{\varepsilon}} \log_{10} \left\{ \frac{d}{2a} \left(1 - \frac{d}{2s}\right)^{1/2} \right\} \text{ ohms}
\]

\[
(3.15)
\]

The form of this expression suggests that \( Z_0 \) is most sensitive to change of radius of the centre conductors; this may be confirmed by substituting the values of \( \frac{d}{2a} \) and \( \frac{s}{a} \) stated on page 45.

**THE PHASE CONSTANT \( \beta \)**

Along with the attenuation constant \( \alpha \), \( \beta \) is given by the relation

\[
P = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}
\]

On squaring both sides, equating real and imaginary parts and eliminating \( \alpha \),
HIGH FREQUENCY TRANSMISSION LINES

\[ \beta = \sqrt{\frac{1}{2} (\sqrt{(R^2 + \omega^2 L^2)}(G^2 + \omega^2 C^2) - (RG - \omega^2 LC))} \] radians/metre

\[ \omega \sqrt{LC} \] at very high frequencies for the reasons discussed in the preceding section.

The wavelength \( \lambda = \frac{2\pi}{\beta} = \frac{1}{j \sqrt{LC}} \), so that the velocity of principal wave propagation in the line, \( \lambda_f \), is \( \frac{1}{\sqrt{LC}} \). To the degree of accuracy associated with neglecting the internal inductance of the conductors relative to the inductance of the dielectric space, the velocity is thus independent of the frequency and, by reference to relations (3.10), (3.13) and (3.14), is equal to

\[ \sqrt{\frac{\varepsilon_0}{\varepsilon}} \times 3 \times 10^8 \text{ metres/sec.} \]

for all forms of line.

THE ATTENUATION CONSTANT \( \alpha \)

Before discussing the form of the attenuation constant \( \alpha \) it is desirable to refer to the 'units' in which a difference in power level between two points in a communication system is normally expressed, since these are the units in which it will be found convenient to express \( \alpha \).

Consider the points \( A \) and \( B \) in the network shown in Fig. 14; the input impedance of the circuit at \( A \) has magnitude \( |Z_i| \) and phase angle \( \phi_1 \), while the impedance to the right of \( B \) is \( |Z_s| |\phi_2| \). If the voltage magnitudes at \( A \) and \( B \) are \( V_1 \), \( V_2 \), the powers entering \( Z_1 \) and \( Z_2 \) are

\[ W_1 = \frac{V_1^2}{|Z_1|} \cos \phi_1 \text{ and } W_2 = \frac{V_2^2}{|Z_2|} \cos \phi_2 \] respectively.

The difference in power level between \( A \) and \( B \) may be expressed either as

\[ \frac{1}{2} \log_2 \left( \frac{W_1}{W_2} \right) \text{ nepers, or, more commonly, } 10 \log_{10} \left( \frac{W_1}{W_2} \right) \text{ decibels, } \]

the relationship between the two units being

\[ 1 \ \text{nepers} = 8.686 \ \text{db.} \]

Written in terms of the voltage and impedance values the difference in power level becomes

\[ 20 \log_{10} \left( \frac{V_1}{V_2} \right) + 10 \log_{10} \left( \frac{|Z_2|}{|Z_1|} \right) + 10 \log_{10} \left( \frac{\cos \phi_1}{\cos \phi_2} \right) \text{ db.} \]

It will be seen from this that the difference becomes

\[ 20 \log_{10} \left( \frac{V_1}{V_2} \right) \text{ if, but only if, the impedances across which the voltage is measured are the same in magnitude and phase.} \]

Now, in the case of the infinitely long line, the impedance at every point along it, expressed as the ratio of the voltage across the line at a point to the current passing through it into the line beyond, is \( Z_0 \). Hence, since the voltage at distance \( x \) from the input end is \( V_x e^{-\alpha x} \), the ratio of the input power to the power available at \( x \) is \( e^{2\alpha x} \) and the difference in power level \( \frac{1}{2} \log_2 (e^{2\alpha x}) = \alpha x \) nepers. The rate of decay of power level is thus \( \alpha \) nepers or 8.686 \( \alpha \) decibels per unit length of line.

On eliminating \( \beta \) from the expression for \( P \),

\[ \alpha = \sqrt{\frac{1}{2} (\sqrt{(R^2 + \omega^2 L^2)}(G^2 + \omega^2 C^2) + (RG - \omega^2 LC))} \] nepers/metre

which reduces, for very high frequencies, to

\[ \alpha = \frac{1}{2} R \sqrt{\frac{C}{L}} + \frac{1}{2} G \sqrt{\frac{L}{C}} = \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right) \] nepers/metre (3.16)

\[ = 4.34 \left( \frac{R}{Z_0} + GZ_0 \right) \text{ db/metre} \]
HIGH FREQUENCY TRANSMISSION LINES

VALUE OF $\alpha$ FOR THE COAXIAL LINE

On substituting for $R$, $G$ and $Z_0$ from (3.10) and (3.15),

$$\alpha = 9.95 \times 10^{-6} \sqrt{f} \sqrt{\frac{1}{\varepsilon a} + \frac{1}{b \varepsilon_0}}$$

$$+ 9.10 \times 10^{-6} \sqrt{\frac{\varepsilon f \tan \delta}{\varepsilon_0}} \text{db/metre} \quad (3.17)$$

in which $f$ is the frequency in cycles per second, $\frac{\varepsilon}{\varepsilon_0}$ is the ratio of the permittivity of the line dielectric to that of air, $a$ and $b$ are in metres and $\sigma_a$ and $\sigma_b$, the conductivities of the inner and outer conductors, are in mhos per metre cube ($5.9 \times 10^7$ for copper, $4.65 \times 10^6$ for lead).

The contribution to the attenuation of the ohmic losses in the conductors thus increases as the square root of the frequency, while that due to dielectric loss increases directly as the frequency. Evidently the latter contribution will become increasingly important the higher the frequency. The order of frequency at which this contribution becomes the dominant one depends on the value of $\tan \delta$.

The following figures calculated for various frequencies up to $10^{10}$ c.p.s. (10,000 Mc/sec.) for a feeder of copper centre conductor $a = 0.05$ cm., lead outer conductor $b = 0.5$ cm. and continuous solid dielectric core of $\frac{\varepsilon}{\varepsilon_0} = 2.3$, $\tan \delta = 5 \times 10^{-4}$, say, demonstrate the importance which attaches to the latter property of dielectric materials at very high frequencies.

<table>
<thead>
<tr>
<th>Frequency (c.p.s.)</th>
<th>Total attenuation db/metre</th>
<th>Percentage contribution of losses in conductor</th>
<th>Percentage contribution of losses in dielectric</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^6$</td>
<td>$5.35 \times 10^{-3}$</td>
<td>73</td>
<td>1</td>
</tr>
<tr>
<td>$10^7$</td>
<td>$1.75 \times 10^{-3}$</td>
<td>71</td>
<td>4</td>
</tr>
<tr>
<td>$10^8$</td>
<td>$5.95 \times 10^{-2}$</td>
<td>65</td>
<td>12</td>
</tr>
<tr>
<td>$10^9$</td>
<td>$2.35 \times 10^{-1}$</td>
<td>52</td>
<td>29</td>
</tr>
<tr>
<td>$10^{10}$</td>
<td>$1.23$</td>
<td>31</td>
<td>57</td>
</tr>
</tbody>
</table>

By way of emphasizing the significance of these results it might be mentioned that at $10^{10}$ c.p.s. more than one-half of the input power would be dissipated as line losses in a 3-metre length of the line considered. The losses in an all-copper and substantially air dielectric line of the same dimensions would, of course, be much smaller—and could be reduced further by increasing the dimensions of the conductors. But even so, as discussed by Lamont,* the coaxial line compares unfavourably with the waveguide in this respect for the propagation of waves of this high order of frequency, for which the guide dimensions have become manageable.

COAXIAL LINE DIMENSIONS FOR MINIMUM ATTENUATION

The form of the first term of equation (3.17) suggests that there is some ratio of $b$ to $a$ for which this term is a minimum. Differentiation shows that this ratio is given by the equality

$$\log_e \frac{b}{a} = 1 + \sqrt{\frac{\sigma_a}{\sigma_b}}$$

For copper centre and outer conductors the optimum $\frac{b}{a}$ value is 3.59, and for copper centre and lead outer, using the $\sigma$ values stated previously, 5.32. It will be found on trial, however, that the magnitude of the conductor loss is not very sensitive to variations of $\frac{b}{a}$ on either side of these values; thus it is within 5 per cent of the minimum value over the range of $\frac{b}{a}$ from 5.3 to 2.6 in the all-copper line. This is fortunate since the ratio $\frac{b}{a}$ is usually determined by the desired value of the characteristic impedance $Z_0$, given, for the coaxial line, by the expression

* Lamont, Wave Guides, p. 54.
The range $5.3-2.6$ for $\frac{b}{a}$ covers a range of $Z_0$ for air-spaced lines from 100 to 57 ohms.

ATTENUATION EXPRESSIONS FOR TWIN LINES

**Unscreened Twin**

$$\alpha = 9.95 \times 10^{-6} \sqrt{\frac{\varepsilon}{\varepsilon_0}} \frac{\sqrt{\varepsilon}}{a \sqrt{\varepsilon_0}} \left( \frac{d}{2a} \right)^2 - 1$$

$$= 9.10 \times 10^{-8} \sqrt{\frac{\varepsilon}{\varepsilon_0}} \cdot f \tan \delta \text{ db/metre}$$

Differentiation of the first term of this expression with respect to $\frac{d}{2a}$ leads to the result that the conductor loss contribution to the attenuation passes through a minimum for $\frac{d}{2a} = 2.27$. These relative dimensions correspond to a characteristic impedance value, for an air-spaced line, of 175 ohms.

**Screened Twin**

$$\alpha = 9.95 \times 10^{-6} \sqrt{\frac{\varepsilon}{\varepsilon_0}} \left( \frac{d}{2a} \right)^2$$

$$+ 9.10 \times 10^{-8} \sqrt{\frac{\varepsilon}{\varepsilon_0}} \cdot f \tan \delta \text{ db/metre}$$

Kirschstein has studied arithmetically the variation of the conductor loss term with the ratios $\frac{d}{2s}$ and $\frac{s}{a}$ and finds that, for copper conductors throughout, the optimum values of these ratios are 0.45 and 5.5 respectively. This corresponds to an air-spaced line of characteristic impedance 145 ohms. For the case of a lead outer the optimum values are $\frac{d}{2s} = 0.37$, $\frac{s}{a} = 6.8$, the corresponding $Z_0$ value then being 161 ohms. The variation of the attenuation is so flat around these values, however, that considerable deviation from them is permissible without serious reaction on the attenuation.

MAXIMUM VALUE OF ELECTRIC STRESS AND OF THE POWER WHICH MAY BE TRANSMITTED THROUGH A LINE

(a) **Coaxial Line.** The electric stress in a coaxial line of conductor radii $a$ and $b$ is greatest at the surface of the centre conductor, and for perfectly smooth surfaces is given in terms of the voltage $V$ between conductors by the expression

$$E = \frac{\varepsilon_0}{\varepsilon} \frac{V}{a \log \frac{b}{a}} \text{ volts/metre}$$

$a$ and $b$ being expressed in metres.

With air dielectric, corona discharge will be initiated at the centre conductor if the electric stress there is permitted to exceed a peak value of about 3,000 kV./metre (30 kV./cm.) at 25°C and 76 cms. mercury. Although this figure has resulted from experimental work at power frequencies, and there are as yet no reliable data for the electric strength of air at frequencies in excess of about 10 Mc/sec., there is reason to suppose that the safe electric stress at very high frequencies is greater, rather than less, than 3,000 kV./metre. In the region of 1 Mc/sec, the electric strength appears to be lower than the 50-cycle figure to
the extent of 10–15 per cent. According to Müller it decreases with increase of frequency up to 7 Mc/sec. but increases beyond until at 25 Mc/sec. it is only about 6 per cent below the power frequency figure.

The maximum power which can be transmitted through a line of given outer conductor radius $b$ without the maximum permissible stress $E_{\text{max}}$ being exceeded at the surface of the centre conductor may be calculated as follows. The power $W$ in an air dielectric line is given by the general expressions

$$W = \frac{V^2}{2Z_0} = \frac{1}{2} \cdot \frac{\left(E_0 \log_{10} \frac{b}{a}\right)^2}{60 \log_{10} \frac{b}{a}} = \frac{1}{120} E_0 b^3 \log_{10} \frac{b}{a} \text{ watts}$$

For specified values of $b$ and $E$ ($= E_{\text{max}}$), this has maximum value for $\frac{b}{a} = 1.65$, corresponding to a line of characteristic impedance 30 ohms. The maximum transmissible power through this line of outer radius $b$ is thus

$$W_{\text{max}} = \frac{b^2 E_{\text{max}}^2}{652} \text{ watts}$$

It will be noted from the above and the discussions on pages 46 and 51 that the most suitable value of the ratio $\frac{b}{a}$ depends on the particular purpose to be served.

In view of the fact that the electric strength of solid dielectrics, and of Polythene in particular, is much greater than that of air, it would seem that stresses considerably in excess of 3,000 kV./metre might be permitted at the

we shall suppose the outer conductor of the line in question to be covered by a protective sheath of outside radius \( r_0 \). The thickness of the outer conductor will be ignored.

If the line is correctly terminated and the input power is \( W_s \) watts, the line current and voltage at the input will be \( \sqrt{W_s/Z_0} \) and \( \sqrt{W_s/a} \) respectively, and the total power loss in the first unit length of line (taken arbitrarily here as one metre) will consist of

1. the loss in the centre conductor
   \[ W_a = R_a \cdot \frac{W_s}{Z_0} \text{ watts} \]

2. that in the outer conductor
   \[ W_b = R_b \cdot \frac{W_s}{Z_0} \text{ watts} \]

3. that in the dielectric between the two conductors
   \[ W_d = G \cdot W_s Z_0 \text{ watts} \]

where \( R_a \) and \( R_b \) are the conductor resistances, and \( G \) the dielectric conductance, per metre length (equation 3.10).

The temperature rise of the centre conductor above ambient \( (\theta_a - \theta_0)^o \text{ C.} \), which results from this power loss will be governed, if longitudinal thermal conduction is ignored, by the radial thermal resistance of the system.

The component parts of this resistance are

1. the thermal resistance of the dielectric core
   \[ S_{ab} = \frac{1}{2\pi} \rho_1 \cdot \log_e \frac{b}{a} \text{ thermal ohms/metre} \]

   where \( \rho_1 \) is the thermal resistivity of the core material.

2. that of the protecting sheath
   \[ S_{bc} = \frac{1}{2\pi} \rho_2 \cdot \log_e \frac{c}{b} \text{ thermal ohms/metre} \]

   where \( \rho_2 \) is the thermal resistivity of the covering.

3. that of the surface of the protecting sheath.

Exact calculation of the resistance external to the covering is not normally possible because the dissipation from the surface depends appreciably on the nature both of the surface and of its environment. However, it is known that for a cable mounted in still air, the thermal dissipation per unit length is given by the empirical relation

\[ W = K A (\theta_a - \theta_0)^\frac{1}{2} \]

in which \( K \) is the emissivity of the surface, \( A = 2\pi r_0 \times \tau \) its area per unit length and \( (\theta_a - \theta_0) \) the excess temperature above ambient. This gives for the effective thermal resistance external to the sheath the expression

\[ S_{ca} = \frac{1}{2\pi K (\theta_a - \theta_0)^\frac{1}{2}} \text{ thermal ohms/metre} \]

The following equations are therefore available for the evaluation of the temperature rise \( (\theta_a - \theta_0) \) of the centre conductor above ambient

\[ \theta_a - \theta_0 = S_{ab} \left( W_a + \frac{W_d}{2} \right) \]

it being assumed that the dielectric loss is concentrated at the geometric centre of the dielectric,

\[ \theta_b - \theta_0 = S_{bd}(W_a + W_d + W_b) \]

\[ \theta_0 = \theta_a = S_{ca}(W_a + W_d + W_b) \]

Since \( S_{ca} \) involves \( (\theta_a - \theta_0) \) the latter must first be determined from (3.18) for \( W = W_a + W_d + W_e \), \( A = 2\pi r_0 \), when \( (\theta_a - \theta_0) \) follows, from the addition of 3.19, 20 and 21, as

\[ (\theta_a - \theta_0) = S_{ab} \left( W_a + \frac{W_d}{2} \right) + (S_{bc} + S_{ca})(W_a + W_d + W_b) \]

The above neglects the effect of increase of conductor resistance and possible change of the dielectric conductance with rise of temperature. The thermal resistivity of Polythene, the material most commonly used for the dielectric core of high-frequency feeders, is of the order of \( 4^0 \text{ C.}/\text{watt/metre cube} \) (400 in centimetre units); that of the covering may be between 5 and \( 20^0 \text{ C.}/\text{watt/metre cube} \).
cube, depending on the material employed. The emissivity constant \( K \) of the outer surface depends on the dimensions of the cable and on the covering material, but is of the order \( 5 - 7 \times 10^{-4} \). Full data on this point and a comprehensive discussion of thermal problems in cables is given in a paper by S. Whitehead and E. E. Hutchings.*


CHAPTER IV

THE BEHAVIOUR OF TERMINATED LINES

It was shown on page 31 that the variations of the voltage and current associated with propagation of the principal wave along a line of uniformly distributed constants are expressed

\[
V_x = Ae^{-pz} + Be^{pz} \\
I_x = \frac{1}{Z_0} [Ae^{-pz} - Be^{pz}] \]

each represented as the sum of forward and backward travelling waves. It was remarked further on page 32 that when the line is supposed of infinite length the backward wave components disappear. This infinitely long line is, of course, hypothetical, but it is possible to terminate a finite length of line in such a manner that it behaves as if of infinite length. The line is then said to be correctly terminated. Under all other circumstances a backward wave component will exist with consequences now to be discussed.

LINE OF LENGTH \( l \) TERMINATED IN AN IMPEDANCE

\[
Z_x = |Z_x| e^{j\theta_x} = \frac{V_x}{I_x} = Z_0 \frac{Ae^{-pz} + Be^{pz}}{Ae^{-pz} - Be^{pz}} \]

The ratio of the voltage to the current at any point along the line is evidently the input impedance of the length of line extending beyond that point to the termination. Thus

\[
Z_x = \frac{V_x}{I_x} = Z_0 \frac{Ae^{-pz} + Be^{pz}}{Ae^{-pz} - Be^{pz}} \quad \ldots (4.1)
\]

However, the connexion of an impedance \( Z_x \) across the line disconnexion at \( x = l \), imposes the condition that at this point \( \frac{V}{I} \) must equal \( Z_x \), and on making this substitution...
and rearranging it follows that
\[
\frac{Be^{pi}}{Ae^{-pi}} = \frac{Z_T - Z_0}{Z_T + Z_0} = k_T \text{ say} \ldots (4.2)
\]
This expression gives the ratio at the termination of the backward to the forward wave components of voltage and may be described as the \textit{Voltage Reflection Coefficient} of the termination. In the general case it will be a complex quantity which we shall write \(k_T = K_T e^{i\phi_T}\), where \(K_T\) expresses the ratio of the amplitude of the reflected to that of the incident voltage and \(\phi_T\) the phase change which occurs on reflection.

Remembering that for lines operated at very high frequencies \(Z_0\) is, for all practical purposes, purely real, it follows that
\[
K_T = \left\{ \frac{(\eta^2 + 1)}{\eta^2 - 1} - 2\eta \cos \theta_T \right\} \ldots (4.3)
\]
and
\[
\phi_T = \tan^{-1}\left\{ \frac{2\eta \sin \theta_T}{\eta^2 - 1} \right\} \ldots (4.4)
\]
where \(\eta = \left| \frac{Z_T}{Z_0} \right|\).

Since the phase of the backward component of current is reversed relative to that of the corresponding voltage component, the \textit{current reflection coefficient} is to be written
\[
-\frac{Z_T - Z_0}{Z_T + Z_0} \ldots (4.5)
\]

The resultant voltage applied to the terminating impedance is the sum of the forward and backward voltage components, and its ratio to the forward one \(Ae^{-pi}\) is
\[
1 + \frac{Be^{pi}}{Ae^{-pi}} = 1 + \frac{Z_T - Z_0}{Z_T + Z_0} = 2Z_T
\]
This may be called the \textit{voltage transmission coefficient}; the corresponding \textit{current transmission coefficient} is easily shown to be \(\frac{2Z_0}{Z_T + Z_0}\).

So far \(A\) and \(B\) have been referred to merely as arbitrary constants, but it is now necessary to consider on what factors they depend and, in particular, how they are related to the voltage and current at the source, \(x = 0\). In doing this we shall automatically dispose of the queries likely to have been raised by the above, namely, what happens to the reflected wave from the line termination when it reaches the source, and why it is that in the expressions for \(V\) and \(I\) only one forward and one backward wave appear to be involved.

In Fig. 16 a generator providing an E.M.F. \(\mathcal{E}_0\) and of internal impedance \(Z_0\) is shown connected to the line input. At the instant of this connexion the voltage applied to the line will be \(\mathcal{E}_0 \cdot \frac{Z_0}{Z_T + Z_0} e^{-pi}\); the reason being that pending the return of a voltage wave reflected from the termination \(Z_T\), the line must appear at the input as of infinite length, its input impedance being the characteristic impedance \(Z_0\). At a time \(t = \sqrt{LC}\) later, a voltage given by
\[
\mathcal{E}_0 \cdot \frac{Z_0}{Z_T + Z_0} e^{-pi}
\]
will reach the termination \(Z_T\) and will produce, according to the preceding, a backward wave. At distance \(x\) from the source, that is at \((l - x)\) from the termination, the reflected voltage will be expressed
\[
k_T \left[ \mathcal{E}_0 \cdot \frac{Z_0}{Z_T + Z_0} e^{-pi} \right] e^{-p(l-x)} = k_T \left[ \mathcal{E}_0 \cdot \frac{Z_0}{Z_T + Z_0} e^{-2pi} \right] e^{px}
\]
On reaching the input end of the line, \( x = 0 \), this wave will be reflected likewise to produce a supplementary forward wave

\[
k_0k_r \left\{ \frac{E_0 \cdot Z_o}{Z_0 + Z_o} e^{-2\pi x} \right\} e^{-Pz}
\]

where \( k_0 \) expresses the voltage reflection coefficient of the input termination. A continuation of this process of successive reflection leads finally to a quasi-stationary condition when the resultant voltage at a point distant \( x \) from the input end of the line is given by

\[
V_x = E_0 \frac{Z_0}{Z_o + Z_a} \left( 1 + k_0k_r e^{-2\pi x} + (k_0k_r e^{-2\pi x})^2 + \ldots \right) e^{-Pz} + E_0 \frac{Z_0}{Z_o + Z_a} k_r e^{-2\pi x} \left( 1 + k_0k_r e^{-2\pi x} + (k_0k_r e^{-2\pi x})^2 + \ldots \right) e^{Pz}
\]

\[
= E_0 \frac{Z_0}{Z_o + Z_a} \left( e^{-Pz} + (k_r e^{-2\pi x}) e^{Pz} \right)
\]

which is of the form \( Ae^{-Pz} + Be^{Pz} \) stated in (3.3).

It is seen that the form of the voltage distribution is fully determined by the first forward and first backward waves—that is, by the reflection coefficient \( k_r \) of the line termination—the effect of the infinite succession of subsequent reflections being contained in the constant factor \( (1 - k_0k_r e^{-2\pi x}) \).

**CONSIDERATION OF LINE DISCONTINUITIES IN TERMS OF MAXWELL’S EQUATIONS**

Before proceeding to an extension of the above treatment we must analyse the restrictions which adherence to the premises of Chapter II imposes on the nature of the impedances which we have represented so conveniently as \( Z_x \) and \( Z_o \).

In order to do this we shall go back to the field equations governing the propagation of a wave the electric and magnetic field components of which are purely transverse to the direction of propagation \( x \), namely

\[
\frac{\partial E}{\partial x} = -\frac{\partial H}{\partial t} \quad ; \quad \frac{\partial H}{\partial x} = -\sigma E - \frac{\partial E}{\partial t} \quad . (2.5) \text{ and (2.6)}
\]

The solution of these equations is of precisely the same form as (3.3) and (3.4). For a wave of frequency \( \frac{\omega}{2\pi} \), and considering only the forward wave solution, the time amplitudes of \( E \) and \( H \) are given by

\[
E = ae^{-pz} ; \quad H = \frac{1}{z_o} \cdot ae^{-pz} \quad . . . \quad (4.7)
\]

in which \( p = \sqrt{j\mu_0(\sigma + j\omega\epsilon)} \) and \( z_o = \sqrt{\frac{j\mu_0}{\sigma + j\omega\epsilon}} \). (4.8)

\( z_o \) denotes the ratio of the electric to the magnetic field intensities and is evidently of the nature of an impedance—it is called the *Wave Impedance*. Where \( \sigma \) is negligible compared with \( \omega \epsilon \), it is given by

\[
z_o = \sqrt{\frac{\mu}{\epsilon}}
\]

and for free space, for which \( \mu_0 = 4\pi \times 10^{-7} \) henrys/metre, 

\[
\epsilon_o = \frac{1}{360\pi} \times 10^{-9} \text{ farads/metre}, \text{ it has the value } 377 \text{ ohms.}
\]

For a dielectric medium of relative permittivity \( \frac{\epsilon}{\epsilon_o} \) its value is \( 377 \sqrt{\frac{\epsilon}{\epsilon_o}} \) ohms.

It will be seen from comparison with (3.5) that it differs from the characteristic impedance \( Z_0 \) of our transmission lines by a purely numerical factor; for the coaxial line this factor is \( \frac{1}{2\pi} \log \frac{b}{a} \).

Suppose now, as depicted in Fig. 17, that this forward wave meets at \( x = l \) along the line a discontinuity consisting of a change of dielectric constants from \( \epsilon_1, \mu_1, \sigma_1 \) to \( \epsilon_2, \mu_2, \sigma_2 \), it being assumed that the discontinuity occurs at a plane surface transverse to the line so that the electric...
and magnetic field components of the incident wave are purely tangential to this surface.

Under all circumstances the tangential components of both $E$ and $H$ on the two sides of the boundary between the media must be the same. Since, however, $\frac{E}{H}$ in the incident wave is given by $z_{01}$ and in the wave transmitted into the second medium by $z_{02}$, this condition cannot be satisfied unless a reflected wave is set up in the first medium, and, moreover, unless the relative phase of the electric and magnetic field components in this reflected wave differs by $\pi$ from that in the incident wave.

Thus if $E_i, H_i; E_r, H_r$ and $E_t, H_t$ represent the field components, at the boundary, of the incident, reflected and transmitted waves respectively, we have

$$E_i = E_t - E_r; \quad H_i = H_t - H_r$$

$$E_i \frac{E_t}{E_r} = z_{01}; \quad H_i \frac{H_t}{H_r} = z_{02}$$

It follows from these relations that

**Coaxial Line**

**Twin Line**

To infinity

\[ E_i = E_t - E_r; \quad H_i = H_t - H_r \]

\[ E_i \frac{E_t}{E_r} = z_{01}; \quad H_i \frac{H_t}{H_r} = z_{02} \]

**TERMINATED LINES**

$$\frac{E_r}{E_i} = \text{reflection coefficient for the electric field}$$

$$= \frac{z_{02} - z_{01}}{z_{02} + z_{01}}$$

$$\frac{H_r}{H_i} = \text{reflection coefficient for the magnetic field}$$

$$= -\frac{z_{02} - z_{01}}{z_{02} + z_{01}}$$

$$\frac{E_t}{E_i} = \text{transmission coefficient for the electric field}$$

$$= \frac{2z_{02}}{z_{02} + z_{01}}$$

$$\frac{H_t}{H_i} = \text{transmission coefficient for the magnetic field}$$

$$= \frac{2z_{02}}{z_{02} + z_{01}}$$

\[ (4.9) \]

**THE SHORT-CIRCUITED LINE**

A particularly important special case of the preceding is that in which the second medium is a metal. In this case the conduction current in it will dominate the displacement one so that the propagation constant becomes

$$\rho = \sqrt{\mu_0 \sigma_0} = (1 + j) \frac{\sqrt{\mu_0 \sigma_0}}{\sqrt{\pi \mu_0 \sigma_0}}$$

and the wave impedance

$$z_{02} = \sqrt{\frac{j \omega \mu_0 \sigma_0}{\sigma_0}} = (1 + j) \sqrt{\frac{\pi \mu_0 \sigma_0}{\sigma_0}}$$

Following the argument of page 36 the electric and magnetic field components will fall to less than 2 per cent of their surface values at a depth, for copper and at 10⁸ c.p.s., of 0.026 mm., while $z_{02}$ differs from zero by only $2.6 \times 10^{-8}$ ohms. The reflection coefficient of the electric field reaching the copper surface along an air dielectric line is thus $-1 + 7 \times 10^{-8}$, which, experimentally, is indistinguishable from $-1$. 
In the unscreened twin line, a mathematically perfect short circuit requires the line to be terminated in a perfectly conducting metal plate extending to infinity in all directions in the transverse plane. Practically, a copper plate of dimensions large compared with the conductor separation will suffice.

OTHER FORMS OF DISCONTINUITY

The expressions (4.2), (4.5), &c., and those given in (4.9) are of identical form, as of course they must be since they are derived from the same basic postulate, namely, that the only waves present in the system are purely transverse ones (principal wave). This assumption is sound for the particular form of discontinuity depicted in Fig. 17, for there the boundary conditions to be satisfied can in fact be satisfied over the whole cross-section in terms of the three waves $E_i, H_i; E_n, H_n$ and $E_r, H_r$ of this type. But it is the only form of impedance discontinuity for which this is the case. It is not the case in the important practical examples shown in Fig. 18, namely, at a change of line dimensions, a 'stub' attachment, an open end, an aerial feed or a line termination consisting of a carbon resistor of the normal rod type.

In order to satisfy the boundary conditions at all points of the cross-section $PQ$ of Fig. 18 (a), for example, it would be necessary to construct mathematically a combination of three principal waves, analogous to those referred to above, and of the $E$ and $H$ modes discussed in Chapter II. Each form of discontinuity constitutes a separate problem and the mathematical difficulties likely to be associated with their complete solution will be obvious.

The important question therefore arises—to what extent is the simple line theory so far developed of value as a means of treating the behaviour of line systems in which such discontinuities as these occur. The answer depends on the relationship between the line dimensions and the frequency at which it is to be operated. It was remarked in Chapter II that the various $E$ and $H$ modes of the coaxial line are the more rapidly attenuated the longer the exciting wavelength relative to the respective critical wavelengths of these modes, and that if the former is large compared with the longest critical wavelength the maximum 'range' of any mode is about $\frac{a + b}{2}$. Beyond this order of distance from one of the discontinuities shown in Fig. 18, only principal waves will be significant and one may then assign to the junction an impedance value such that the behaviour of the actual line at adequate distance from the junction is exactly that of an idealized line, in which only principal waves exist throughout, terminated in this impedance.

In other words, whenever reference is made later to a
line which is terminated at a particular frequency or wavelength in an impedance $Z_r$, it should be remembered that this impedance value relates only to the principal wave distributions of voltage and current along the line. It must not be supposed therefore—in general—that one is entitled to associate this impedance value with the terminating device in question should it be detached from this line and attached to another of different dimensions, even though the characteristic impedances of the two lines may be the same. Clearly this is a troublesome complication. The treatment of practical high-frequency transmission lines, such as those included in the diagrams of Chapter I, in terms of equations (3.1), and of their solutions (3.3) and (3.4), is indeed only an approximation, and this should be borne in mind throughout the discussion which follows. However, provided that the operating wavelength is large compared with the transverse dimensions of the line the treatment is sufficiently reliable for practical purposes, while where this is not the case it affords a valuable guide, to which there is no simple alternative.

The above argument may be illustrated by the case of the open-ended line of Fig. 18 (c). The open end may often be represented with adequate accuracy as an infinite impedance. On the other hand, the following experimental results show that an invariable assumption to this effect is quite unjustifiable. They show at the same time that no such restriction applies to the representation of a short-circuited end as an impedance of zero value.*

The positive sign in column (A) signifies that with the end open the line behaved like an idealized line longer in length than the actual by the amounts stated. This corresponds to an effective capacitive reactance. The zeroes and infinities in the last four columns are not to be taken literally; these figures mean that to the accuracy of measurement the quantities concerned were respectively very small and very large.

* Refer back to page 65.

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**The 'Correctly-Terminated' Line**

(a) Resistive Disk Termination. The danger associated with an uncritical acceptance of the transmission line equations (3.1), and of their solutions, is brought out perfectly by a consideration of one of the methods of producing a 'correctly' terminated line. It follows from these solutions that if a line is terminated in an impedance equal to its characteristic impedance $Z_0$ no reflected wave will be generated at the termination. Now at very high frequencies $Z_0$ is purely resistive, so that apparently all that is required is to produce a pure resistance of this value in such a form that its attachment to the end of the line does not disturb the transverse nature of the field distribution. Ideally this may be done for the coaxial line by closing the end of the line with a disk resistor, as shown in

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* See page 75 onwards.
The resistance of the disk between the two conductors is given by

\[ R = \frac{1}{2\pi \alpha h} \log_e \frac{b}{a} \ \text{ohms} \quad (4.10) \]

where \( \alpha \) is the conductivity of the material and \( h \) its thickness. Clearly, by suitable choice of \( \alpha \) and \( h \) this resistance can be made equal to the characteristic impedance of the line \( Z_0 = 60 \log_e \frac{b}{a} \ \text{ohms} \).

Disk resistors consisting of a film of graphite supported on a thin disk of bakelized paper and having d.c. resistances of the order of 70-80 ohms are now available commercially so that the behaviour of the above arrangement can be checked experimentally. For this purpose the coaxial line referred to in the preceding table, \( Z_0 = 76 \ \text{ohms} \), was terminated in such a disk of d.c. resistance 77 ohms. The standing wave observed along the line at the wavelength \( \lambda \) for which \( \frac{\lambda}{a(a + b)} = 2 \) is shown in Fig. 20, curve (a), from which it will be seen that the disk is far from being a correct termination. It behaves, in fact, as an impedance of value \( (23.1 - j20.1) \ \text{ohms} \), equivalent to a resistance of 40.5 ohms shunted by a capacitive reactance of 46.5 ohms.

A plausible explanation of this result comes to mind at once; namely, that the resistance of the graphite film at the frequency in question may be substantially different from the d.c. value and that the disk resistor and its mount-

\[ E_z = ae^{-\nu z} + be^{\nu z}; \quad H_z = \frac{1}{z_{\text{in}}} \{ae^{-\nu z} - be^{\nu z}\} \quad (4.11) \]
equations analogous to \((3.3)\) and \((3.4)\). In consequence of the conductivity of the graphite, \(p_2\) and \(\sigma_2\) may be written

\[
\sqrt{j\omega \mu_2 \sigma_2} \quad \text{and} \quad \sqrt{\frac{j\omega \mu_2}{\sigma_2}}
\]

respectively.

We will now make the assumption that \(H_2\) is zero at the outside face \(MN\) of the film, that is at \(x = h\). In this event

\[
ae^{-px} - be^{-px} = 0
\]

and on substitution in equations \((4.11)\),

\[
E_2 = 2ae^{-px} \cosh p_2(h - x) ; \quad H_2 = \frac{2a}{\varepsilon_0 h} e^{-px} \sinh p_2(h - x)
\]

These give as the ratio of \(E\) to \(H\) at the inside face, \(x = 0\),

\[
\frac{E_2}{H_2} = \sqrt{\frac{j\omega \mu_2}{\sigma_2}} \coth \sqrt{j\omega \mu_2 \sigma_2} \cdot h
\]

The requirement that there shall be no reflected wave on the line side of this face is therefore satisfied for

\[
\sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{j\omega \mu_2}{\sigma_2}} \coth \sqrt{j\omega \mu_2 \sigma_2} \cdot h
\]

\[
= \frac{1}{\sigma_2 h}
\]

if, as will be the case, \(\sqrt{j\omega \mu_2 \sigma_2} \cdot h\) is small.

Now for an air dielectric line \(\sqrt{\frac{\mu_1}{\varepsilon_1}} = 377\) ohms. There will therefore be no reflection if the product \(\sigma_2 h\) is such that \(\frac{1}{\sigma_2 h} = 377\) ohms. From \((4.10)\), however, the resistance of a film having this value of \(\sigma_2 h\) is

\[
R = 60 \log \frac{b}{a} \text{ ohms}
\]

the expression stated on page 46 for the characteristic impedance \(Z_e\) of an air line of the dimensions shown in Fig. 19.

This result was to be expected, and the only point out-

standing is the validity of the assumption that \(H\) is zero at the back face \(MN\) of the resistive film. The experimental results on the open-ended line previously recorded suggest that this is not a justifiable assumption. On the other hand, it is made absolutely sound, whatever the wavelength, by the attachment beyond the film of a short-circuited line of length \(\frac{\lambda}{4}\).

The physical separation between the graphite film and the short-circuiting plate when curve \((b)\) of Fig. 20 was taken was rather less than \(\frac{\lambda}{4}\); actually it was 0.88\(\frac{\lambda}{4}\) corresponding to an effective shunt reactance for the disc of about 400 ohms capacitive. This agreed closely with the value calculated from its estimated self-capacitance. In addition, the shunt reactance of the mounting which it was necessary to provide behind the disk in the assembly of Fig. 19 \((a)\) was estimated to be of the order of 500 ohms. The resultant of these, about 220 ohms, is obviously quite incapable of accounting for the impedance value of the termination to which curve \((a)\) of Fig. 20 corresponds. Nor is this accounted for by the simple association of the shunt reactance estimated above with that of the open-ended line at this wavelength as given in the table on page 69. The behaviour of the disk termination is evidently complex and is only reduced to the desired simplicity by the connexion of the short-circuited extension line.†

Further evidence of this is provided by the results of

* See p. 93.

† The content of this section forms part of a paper entitled \"The Solution of Transmission Line Problems by use of the Circle Diagram of Impedance,\" by L. G. H. Huxley and the present author, which was read before the Radio Section of the Institution of Electrical Engineers in February, 1944. In a contribution to the discussion Dr. E. B. Moullin presented an alternative treatment of the resistive disk termination. The full text of the discussion, and the paper itself, will be found in the \textit{Journal of the I.E.E., 1944, 91, Part III, p. 16.}
a series of measurements on the disk-terminated line at the shorter wavelength given by \( \lambda/[\pi(a+b)] = 0.6 \). At this wavelength the termination behaved as an impedance of value \( 43.1 - j5.1 \) ohms (standing-wave ratio 1.77), a value which bears no obvious relationship to the value of \( 23.1 - j20.3 \) ohms obtained at the longer wavelength for which \( \lambda/[\pi(a+b)] = 2.0 \). Nevertheless, the disk was again converted to a ‘correct’ termination (standing-wave ratio 1.02) by the addition of the extension line and suitable adjustment of the short-circuiting piston.

(b) Other Forms of Correct Termination. The resistive disk termination discussed above has no great practical significance since it will usually be necessary to make better use of the power available than merely to dissipate it in a carbon resistor. Nevertheless, for reasons discussed in Chapter I, it is of considerable practical importance that the line termination, incorporating the device to which it is desired to supply power, shall be as non-reflecting as possible. To this end it is usually necessary to make attachments to the feeding-line in front of the ‘load’—the stub of Fig. 18 (b), fitted with a sliding, short-circuiting piston, is one form of attachment—the aim being to produce one or more reflected waves additional to that generated at the load itself and to arrange that the principal wave components of the several reflected waves shall have zero resultant in the feeding line. In the vicinity of such a termination there will clearly be great complexity of field distribution, but, under circumstances discussed previously, this may be restricted in the feeding-line to a very short distance in front of the first discontinuity. Following the argument of page 20 and concerning ourselves only with such principal wave components as exist in this line at adequate distance from the termination, it may then be argued that the complex impedance representing the load has been transformed by the attachments into the correct terminal impedance for the line, namely, its characteristic impedance. This is spoken of as Impedance Transformation and the above process is known as that of Matching the load to the feeding-line.

The Standing Wave Representation of Voltage

The expression for \( V_x \) given in (4.6) may be rewritten

\[
V_x = Ae^{-j\frac{\beta y}{2} + jK_x e^{j\phi_x} e^{-j\frac{\beta y}{2}}}
\]

in which \( y = l - x \) is the distance of the point of observation from the termination \( Z_y \).

For the purpose now intended it is proposed to assume, for simplicity, that the line is devoid of loss in both the conductors and the dielectric. This is justifiable since the effects to be considered are all observable within a distance of about half a wavelength from the termination and we shall be concerned only with relative voltage values within this short length of line. If, then, the attenuation coefficient \( \alpha \) is assumed zero and \( P \) is replaced by \( j\beta \) the above expression for \( V_x \), which we may now call \( V_y \), becomes

\[
V_y = A e^{-j\left(\beta l - \frac{\phi_x}{2}\right)} \left(e^{j\left(\beta y - \frac{\phi_x}{2}\right)} + K_x e^{-j\left(\frac{\phi_x}{2}\right)}\right)
\]

\[
= A \left((1+K_x)\cos\left(\beta y - \frac{\phi_x}{2}\right) + j(1-K_x)\sin\left(\beta y - \frac{\phi_x}{2}\right)\right) e^{-j\left(\beta l - \frac{\phi_x}{2}\right)}
\]

It will be remembered that \( V_y \) is the amplitude, with respect to time \( t \), of a voltage which is varying with time at angular frequency \( \omega \). It is now seen that its real and imaginary parts are periodic functions of \( y \), the natural result of interference between the forward and backward travelling waves discussed in preceding sections. The above expression represents in fact a standing, or stationary, wave of voltage, the modulus of which is given as a function of \( y \) by

\[
|V_y| = A \sqrt{(1+K_x)^2 \cos^2\left(\beta y - \frac{\phi_x}{2}\right) + (1-K_x)^2 \sin^2\left(\beta y - \frac{\phi_x}{2}\right)}
\]

(4.12)
It is this voltage value, or something proportional to it, which is recorded in experimental studies of the voltage distribution along transmission lines.

For values of \( \beta \gamma - \frac{\phi_r}{2} \) equal to \( n \pi \), where \( n \) is a positive integer or zero, this expression attains a maximum value \((1 + K_r)A\), while for values of \( \beta \gamma - \frac{\phi_r}{2} = (2n + 1)\frac{\pi}{2} \), it passes through the minimum value \((1 - K_r)A\). These maxima and minima are separated successively by distances given by \( \beta \gamma = \frac{\pi}{2} \), that is, by \( y = \frac{\pi}{2} \frac{\lambda}{2\pi} = \frac{\lambda}{4} \). The position of the first maximum relative to the termination is given by \( \beta \gamma = \frac{\phi_r}{2} \), that is, by \( y = \frac{\phi_r}{4} \).

Since the reflection coefficient of current differs from \( K_r \) only in sign, the modulus of the current is expressed

\[
|I_v| = \frac{A}{Z_0} \sqrt{(1 - K_r)^2 \cos^2 \left(\beta \gamma - \frac{\phi_r}{2}\right)} + (1 + K_r)^2 \sin^2 \left(\beta \gamma - \frac{\phi_r}{2}\right)
\]

The current maxima thus occur at the points of minimum voltage and vice versa. Moreover, the ratios \( \frac{V_{\text{max}}}{V_{\text{min}}} \) and \( \frac{I_{\text{max}}}{I_{\text{min}}} \) are equal. This ratio of maximum to minimum value of voltage (or current) magnitude is called the Standing Wave Ratio. We shall denote it by \( \rho \).

An important deduction is that \( \frac{V_{\text{max}}}{I_{\text{max}}} \) and \( \frac{V_{\text{min}}}{I_{\text{min}}} \) are each equal to the characteristic impedance \( Z_0 \), irrespective of the reflection coefficient \( K_r \).

**PARTICULAR CASES**

For the ideal short-circuit termination, \( Z_r = 0 \), \( K_r = 1 \) and \( \phi_r = \pi \). In this case the first voltage maximum occurs at \( \frac{\lambda}{4} \) from the termination and the first current maximum at the short circuit itself. The distribution of the voltage and current magnitudes are shown in Fig. 21 (a).

It follows from the above-mentioned deduction that the
current flowing in the short circuit is equal to \( \frac{1}{Z_0} \) times the voltage existing at a distance of \( \frac{\lambda}{4} \) from the termination.

For the idealised open circuit termination, \( Z_r = \infty \) \( K_r = 1 \) and \( \phi_r = 0 \), so that maximum voltage occurs at the open-circuit point and the distribution takes the form shown in Fig. 21 (b).

When the line is correctly terminated, \( Z_r = Z_0 \), and \( K_r = 0 \). There is then no standing wave and the standing wave ratio \( \rho \) is unity.

The behaviour in other cases is illustrated in Fig. 21 (d), (e), (f) and (g).

Determination of the Reflection Coefficient of a Termination (or Other Discontinuity) from the Standing Wave Pattern

From what has been said above, the standing wave ratio and the modulus of the voltage reflection coefficient of the line termination are related by the expression

\[
\rho = \left| \frac{V_{\text{max}}}{V_{\text{min}}} \right| = \frac{1 + K_r}{1 - K_r}
\]

Experimental determination of \( \rho \) thus provides the value of \( K_r \) through the relation

\[
K_r = \frac{\rho - 1}{\rho + 1} \quad \cdots \cdots \cdots \quad (4.13)
\]

The phase angle of the reflection coefficient follows from the values of voltage magnitude at \( y = \frac{\lambda}{2} \) and \( y = \frac{\lambda}{4} \). On substituting these values of \( y \) in (4.12)

\[
\left| \frac{V_{\lambda/2}}{V_{\lambda/4}} \right|^2 = \frac{1 + K_r^2 + 2K_r \cos \phi_r}{1 + K_r^2 - 2K_r \cos \phi_r}
\]

from which \( \phi_r \) is determinate since \( K_r \) is known from (4.13).

The latter calculation necessarily identifies the termination, or discontinuity, with a particular point—the point taken as \( y = 0 \) in the preceding. Such a point can always be selected as the basis of evaluation, and this having been done it must not be forgotten that the derived \( \phi_r \) is related specifically to this point. In the system of Fig. 18 (a) one would naturally choose as reference the point at which the change of section occurs; in that of Fig. 18 (b), the centre line of the stub entry; in Fig. 12 (c) and (d), the point of line disconnexion. In more complex systems, the selection of a point of reference would be a more arbitrary matter.

This consideration is a very important one when the attempt is made to attach an impedance value to the termination. Before proceeding to a discussion of this matter,* however, mention must be made of the means by which the standing wave pattern may be delineated.

It will have been noted that the evaluation of \( K_r \) in magnitude and phase from (4.13) and (4.14) involves voltage ratios, and not the absolute value of the voltage. This is fortunate since the quantitative determination of voltage at very high frequency offers great difficulty. Evidently the required purpose will be served by observing relative values of the electric field strength from point to point along the line. This procedure cannot be carried out, of course, when the dielectric is a solid one, but Fig. 22 illustrates how it may be performed in an air-dielectric coaxial line. It shows a movable probe penetrating the dielectric space and connected through a crystal or thermocouple detector to a galvanometer. The first requirement of such an arrangement is that it must provide adequate galvanometer deflection at the minimum of the standing wave pattern, but that the probe penetration employed for this purpose must be as small as possible with a view to ensuring minimum disturbance of the field distribution. Excessive coupling between the detector and the line shows itself in distortion of the wave pattern and

* See next section.
inaccuracy in the computations from this pattern. Considerable difficulty in this connexion may be experienced at the highest frequencies where the power available is likely to be small, and it is desirable to check the accuracy of recording by performing measurements with the measuring-line terminated in a short-circuiting plate.

![Diagram of standing wave detector](image)

**Fig. 22.**—Standing wave detector.

If the latter measurements reveal inaccuracy, a number of possible additional causes should be taken into consideration. One is that the slide may be imperfect so that the degree of probe penetration varies over the travel, a second that there may be radiation from the slot due to inadequate shielding, a third, where a crystal detector is employed, that the characteristic may not be truly square law over the whole range of field strength to be measured, and a fourth the presence of harmonics in the source.

**Determination of an Impedance Value to Represent the Behaviour of a Line Termination**

*The 'Brückmann' Method of Impedance Measurement.*

For this purpose we shall revert to equation (4.6), in which

$$k = \frac{Z_r - Z_0}{Z_r + Z_0} \quad \text{and} \quad k_0 = \frac{Z_0 - Z}{Z_0 + Z}$$

On making these substitutions and re-arranging, equation (4.6) becomes

$$V_x = E_0 Z_0 \cdot \frac{Z_0 \sinh P(l-x) + Z_r \cosh P(l-x)}{[Z_0 \{Z_0 \sinh P(l) + Z_r \cosh P(l)\} + Z_0 \{Z_0 \cosh P(l) + Z_r \sinh P(l)\}]} \quad (4.15)$$

But on putting \(x = 0\) in (4.1), the input or sending end impedance of the line, \(Z_0\), is seen to be given by

$$Z_s = \frac{Z_0 A + B}{A - B}$$

$$= \frac{Z_0}{Z_0 + Z_r} e^{-2P_l}$$

from (4.2)

$$= \frac{Z_0 \sinh P_l + Z_r \cosh P_l}{Z_0 \cosh P_l + Z_r \sinh P_l} \quad (4.16)$$

$$= \frac{Z_0}{Z_0 + Z_r} \tan P_l$$

so that (4.15) may be re-written

$$V_x = \frac{E_0 Z_0}{Z_0 + Z_r} \cdot \frac{Z_0 \sinh P l - x + Z_r \cosh P l - x}{Z_0 \sinh P l + Z_r \cosh P l}$$

or

$$V_x = \frac{E_0 Z_0}{Z_0 \sinh P l + Z_r \cosh P l} \quad (4.18)$$

in which \(V_x\) is the input, or sending end, line voltage, and
The analogous current expression is

\[ I_y = \frac{V_y}{Z_0} \frac{Z_0 \cosh Py + Z_T \sinh Py}{Z_0 \sinh Pl + Z_T \cosh Pl} \]  \hspace{0.5cm} (4.19)

These expressions for \( Z_y, V_y \) and \( I_y \) could, of course, have been derived directly—and more simply—from equations (3.3) and (3.4) without invoking reflection coefficients. The latter are of such fundamental significance, however, that it was considered desirable to introduce them \textit{ab initio}.

The voltage \( V_T \) at the termination \( y = 0 \) is

\[ V_T = \frac{V_y}{Z_0} \frac{Z_T}{\sinh Pl + Z_T \cosh Pl} \]  \hspace{0.5cm} (4.20)

so that (4.19) may be expressed

\[ V_y = V_T \left\{ \cosh Py + \frac{Z_0}{Z_T} \sinh Py \right\} \]  \hspace{0.5cm} (4.21)

Since the immediate aim is similar, we shall now make the same approximation as in the preceding section, namely, that of neglecting line losses and replacing \( P \) by \( j\beta \). Equation (4.21) then becomes

\[ V_y = V_T \left\{ \cos \beta y + j \frac{Z_0}{Z_T} \sin \beta y \right\} \]  \hspace{0.5cm} (4.22)

Replacing \( Z_T \) by \( R_T + jX_T \), and treating \( Z_0 \) as purely real, it follows that

\[ \left| \frac{V_y}{V_T} \right|^2 = \frac{\left| Z_T \right|^2 + Z_0^2}{2 \left| Z_T \right|^2} \left[ 1 + \frac{1}{\left( \frac{2R_T Z_0}{\left| Z_T \right|^2 + Z_0^2} \right)^2} \sin (2\beta y + \gamma) \right] \]

where

\[ \gamma = \tan^{-1} \frac{\left| Z_T \right|^2 - Z_0^2}{2X_T Z_0} \]

Now note from (4.22) that when \( \sin \beta y = 0 \), \( \left| \frac{V_y}{V_T} \right| = 1 \)

\* The reader should satisfy himself that this is the same function of \( y \) as that given in terms of the reflection coefficient \( K_T \) in (4.12).
from which it follows that
\[
\cos \theta_f = \frac{\left( |V_{\text{max}}| + |V_{\text{min}}| \right)^2 - |V_{\lambda/2}|^2 - |V_{\lambda/4}|^2}{2 \left| V_{\lambda/2} \right| \left| V_{\lambda/4} \right|}. \tag{4.24}
\]

The magnitude of the terminating impedance referred to the selected reference point \( y = 0 \) is therefore given in terms of \( Z_0 \) by the ratio of the voltages measured at distances \( \lambda/2 \) and \( \lambda/4 \) from this point, and its phase angle is determinate from these two voltages and the maximum and minimum voltage values.

An alternative expression for \( \theta_f \) is
\[
\cos \theta_f = \frac{\left| V_{\lambda/2} \right|^2 + \left| V_{\lambda/4} \right|^2 - \left( \left| V_{\text{max}} \right| - \left| V_{\text{min}} \right| \right)^2}{2 \left| V_{\lambda/2} \right| \left| V_{\lambda/4} \right|}. \tag{4.25}
\]

while a still further and more convenient form is obtained on adding (4.24) and (4.25), namely
\[
\cos \theta_f = \frac{\left| V_{\text{max}} \right| \left| V_{\text{min}} \right|}{\left| V_{\lambda/2} \right| \left| V_{\lambda/4} \right|}. \tag{4.26}
\]

Formulae (4.24) and (4.25) were first derived by Brückmann,* and this technique of impedance measurement is frequently referred to as the Brückmann method.

The sign of \( \theta_f \) and therefore of \( X_f \) may be deduced from observation of the position of the first voltage minimum relative to the termination. Thus—see Fig. 21 (f)—with \( Z_\tau \) inductive, \( X_\tau \) and \( \theta_\tau \) positive, the first voltage minimum will lie more distant than \( \lambda/4 \) and less distant than \( \lambda/2 \) from the termination, while—see Fig. 21 (g)—with \( Z_\tau \) capacitive, \( X_\tau \) and \( \theta_\tau \) negative, it will lie within the region \( y = 0, \ y = \lambda/4 \).

An alternative expression for \( \theta_\tau \) which is more reliable than (4.26) for the evaluation of small phase angles, may be derived from a paper by Hempel.* It is
\[
\sin \theta_\tau = \frac{1}{2} \left[ \frac{\left| V_{\lambda/2} \right|}{\left| V_{\lambda/4} \right|} - \frac{\left| V_{\lambda/4} \right|}{\left| V_{\lambda/2} \right|} \right] \tan \frac{\pi}{\lambda} \left( y_{\text{min}} - \frac{\lambda}{4} \right). \tag{4.27}
\]

where \( y_{\text{min}} \) denotes the position of the first minimum voltage magnitude.

It will be evident from the above that the impedance value for a termination, or for whatever form of line discontinuity is involved, depends fundamentally on the decision taken as to the point \( y = 0 \) at which the termination is to be regarded as located. Furthermore, the effective impedance of a device may be affected very considerably at very short wavelengths by the minor details of the connexion, so that wherever possible the measurement should be performed on an assembly identical in all significant respects with that to be employed in practice. This is, of course, not always convenient, and impedance measuring-sets operating on the principle discussed above have been developed and used extensively. In their early work at 100 and 600 Mc/sec, respectively, Brückmann and Hempel used open wire lines. They did not connect the impedance to be measured at the end of the line, but at a point \( \lambda/4 \) from the end which they short-circuited. This method is not very practicable at very high frequencies, where it is preferable from several points of view to employ slotted coaxial lines, Fig. 22, with the device to be measured connected to one end.

POWER MEASUREMENT FROM THE STANDING WAVE PATTERN

It follows from the preceding that
\[
\frac{\left| V_{\text{max}} \right| \left| V_{\text{min}} \right|}{\left| V_\tau \right|^2} = \frac{Z_0 R_\tau}{|Z_\tau|^2}
\]


* W. Hempel, E.N.T., 1937, 14, p. 33.
that is, that \[ \frac{|V_{\text{max}}|}{V_{\text{min}}} = R_f \left| \frac{V_f}{Z_f} \right|^2 = R_f |I_f|^2 \]

= power absorbed in the termination.

Provided, therefore, that the measuring-line can be calibrated so that the indications on the probe, Fig. 22, give a quantitative measure of voltage, the power may be derived from the maximum and minimum voltage values and the supposed known \( Z_0 \) of the line, whatever the magnitude and phase angle of the load impedance.

POWER TRANSFER FROM THE GENERATOR TO THE TERMINATING IMPEDANCE

(a) Power Input to the Line. The transmission line system drawn in Fig. 16 may be replaced by the equivalent circuit of Fig. 23, in which \( Z_s \) denotes the input impedance of the line

\[ Z_s = Z_0 \left( \frac{Z_f}{Z_0} + \tanh Pl \right) \]

and \( V_s \) the input voltage

\[ V_s = \frac{E_0}{Z_0 + Z_s} \]

It is easily shown that the power supplied to \( Z_s \) is a maximum, of value \( \frac{E_0^2}{4R_s} \), when \( Z_s \) and \( Z_0 \) are conjugate impedances, and, since \( Z_s \) is a function of both the line length \( l \) and the terminating impedance \( Z_f \), any variation of either of the latter will react in some degree on the power input to the line.

This simple representation of the behaviour of a high-frequency oscillator is, however, of very questionable validity. The connexion of a load to such an oscillator is dealt with briefly in Chapter I, and it will be supposed in what follows that, on the completion of adjustments at the termination, steps will be taken at the generator to ensure maximum possible power input to the line. This point understood, we shall consider the simplified system of Fig. 24 in which \( V_s \) denotes the line input voltage which results from the above procedure.

(b) Efficiency of the Line Transmission. It will now be convenient to write

\[ \frac{Z_f}{Z_0} = r_f + jx_f = \tanh (u + jv) = \tanh \psi \]

On making this substitution in the expression (4.17) for \( Z_s \), the latter may be written

\[ Z_s = Z_0 \tanh (Pl + \psi) \]

\[ = \frac{\sinh 2(\alpha l + u) + j \sin 2(\beta l + v)}{\cosh 2(\alpha l + u) + \cos 2(\beta l + v)} \]

FIG. 24.
The input power to the line is then
\[ W_s = \frac{V_s^2}{|Z_s|^2} \times \text{real part of } Z_s \]
\[ = \frac{V_s^2}{Z_0} \left( \tanh (Pl + \psi) \right)^2 \left( \cosh 2(\alpha l + u) + \cos 2(\beta l + v) \right) \]

Similarly, by use of expression (4.20) for \( V_T \), the power supplied to the terminating impedance is given by
\[ W_T = \frac{V_T^2}{Z_0} \left( \sinh (Pl + \psi) \right)^2 \frac{\cosh 2u}{\sinh 2(\alpha l + u)} \cdot \cdots \cdots \cdots \cdot (4.29) \]

on re-arranging.

For the particular case in which \( Z_T = Z_0 \), the line system behaves as if it were part of an infinite line and \( W_T = e^{-2\alpha l} \).

Under these circumstances the efficiency of power transfer to the termination is a maximum and we shall write \( e^{-2\alpha l} = \eta_{\text{max}} \). The efficiency for other terminating impedances is then given by
\[ \frac{\eta}{\eta_{\text{max}}} = \frac{2}{(1 + \coth 2u) + (1 - \coth 2u)\eta_{\text{max}}} \cdot (4.30) \]

This may be put in an alternative form for purely resistive terminations. In this case \( Z_T = \frac{R_T}{Z_0} = \tanh u \), and is equal to the standing wave ratio \( \rho \) if \( R_T \) is greater than \( Z_0 \) and to \( 1 \) if \( R_T \) is less than \( Z_0 \). (4.30) then becomes
\[ \frac{\eta}{\eta_{\text{max}}} = \frac{(\rho + 1)^2 - (\rho - 1)^2}{4\rho \cdot \eta_{\text{max}}} \cdot (4.31) \]

* See Fig. 21.

An important—though rather self-evident—conclusion which follows from either (4.24) or (4.25) is that if there is no line loss, \( \alpha = 0 \), the efficiency of power transfer to the terminating impedance is unity irrespective of the value of this impedance relative to \( Z_0 \) and of the standing wave for which it may be responsible. In other words, 'incorrect' termination \( (Z_T + Z_0) \) is unimportant from the transmission efficiency standpoint in the (hypothetical) loss-free line. In practical lines the transmission efficiency decreases with increasing standing wave ratio due to the increasing significance of the associated line losses. On the other hand, in lines of low overall attenuation coefficient \( \alpha l \) and high maximum efficiency appreciable standing waves can be permitted without significant effect on the transmission efficiency. This will be evident from the data tabulated below relating to a line of overall attenuation 1 db. for which \( \eta_{\text{max}} = 0.795 \). The calculations are made for resistive load.

<table>
<thead>
<tr>
<th>Standing wave ratio ( \rho )</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta / \eta_{\text{max}} )</td>
<td>1.0</td>
<td>0.985</td>
<td>0.955</td>
<td>0.925</td>
<td>0.89</td>
<td>0.83</td>
</tr>
</tbody>
</table>

As discussed on page 3, however, there are other reasons why the removal of appreciable standing waves on feeders is usually desirable.
CHAPTER V

RESONANT LINES

RESONANCE ANDANTI-RESONANCE

Consider a loss-free coaxial line which is short-circuited at both ends and which is energized, as shown in Fig. 25, by means of a loop penetrating into the dielectric space between the conductors. On making the substitutions $\psi = \dot{\omega} = \xi = -1$ and $P = j\beta$ in equation (4.6), this may be written

$$V_x = E_o \cdot \frac{2je^{-j\beta l}}{1 - e^{-j2\beta l}} \sin \beta (l - x) = E_o \cdot \frac{\sin \beta (l - x)}{\sin \beta l}$$

We are already aware of the sinusoidal variation of the voltage with distance $x$ along the line. The present interest is with the magnitude of the voltage as a function of the line length $l$. Two cases are of special interest:

(a) When $l$ is such that $\beta l = n\pi$, i.e. $l = n\frac{\lambda}{2}$, where $n$ is an integer.

In this case $\sin \beta l = 0$ and $V_x$ is infinite, except at $x = l$ where it is zero and at $x = 0$ where it is $E_o$, the reason being that for such single or multiple half-wave short-circuited lines the successive reflected waves from the two ends are all in phase and build up continuously. Under these circumstances the line is said to be Resonant.

(b) When $l$ is such that $\beta l = (2n + 1)\frac{\pi}{2}$, i.e. $l = (2n + 1)\frac{\lambda}{4}$.

In this case $\sin \beta l = \pm 1$ and $V_x = \pm E_o \cdot \cos \beta x$, and thus nowhere exceeds $E_o$. The reason is that mutual cancellation occurs between the successive reflected waves. Under these circumstances the line is said to be Anti-Resonant.

Similar arguments show that lines of length $(2n + 1)\frac{\lambda}{4}$ of which one end is shorted and the other open-circuited, are resonant, and lines of length $n\frac{\lambda}{2}$ anti-resonant.

The form of the voltage and current distributions along the resonant half-wave short-circuited line is shown in Fig. 26 (a), and in the resonant quarter-wave open-circuited line (idealized) in Fig. 26 (b). In consequence of effects discussed in pages 62-9, the resonant lengths of the open-ended line will not be $(2n + 1)\frac{\lambda}{4}$ precisely, nor will the voltage distribution at points close to the open end be of the simple form expressed above.

THE EFFECT OF LINE LOSSES ON THE RESONANT VALUE OF LINE VOLTAGE

Although the expression for $V_x$ given in (4.6) is valuable for purposes of clarification, the alternative form of (4.18)
is more generally convenient. On putting \( Z_T = 0 \) and \( Z_T = \infty \) in this expression in turn, it follows that the voltage distribution along the short-circuited line is

\[
V_y = V_s \frac{\sinh \beta y}{\sinh \beta l}
\]

and along the open-circuited line (idealized)

\[
V_y = V_s \frac{\cosh \beta y}{\cosh \beta l}
\]

\( y \) being measured from the termination.

For small line losses these may be written respectively

\[
V_y = V_s \frac{\alpha \cos \beta y + j \left( 1 + \frac{\alpha^2 \beta^2}{2} \right) \sin \beta y}{\alpha \cos \beta l + j \left( 1 + \frac{\alpha^2 \beta^2}{2} \right) \sin \beta l}
\]

and

\[
V_y = V_s \frac{\left( 1 + \frac{\alpha^2 \beta^2}{2} \right) \cos \beta y + j \alpha \beta \sin \beta y}{\left( 1 + \frac{\alpha^2 \beta^2}{2} \right) \cos \beta l + j \alpha \beta \sin \beta l}
\]

The dominant terms in the two denominators are \( j \sin \beta l \) and \( \cos \beta l \), and there is no significant error in writing as the conditions for resonance in the two cases, \( \sin \beta l = 0 \) and \( \cos \beta l = 0 \) respectively.

Thus for the resonant half-wave short-circuited line

\[
| V_y | = V_s \frac{\sin \beta y}{\alpha \frac{\lambda}{2}}
\]

and for the resonant quarter-wave open-ended line (idealized)

\[
| V_y | = V_s \frac{\cos \beta y}{\alpha \frac{\lambda}{4}}
\]

except very close to the input \( y = l \) where \( | V_y | \) reduces to \( V_s \) in both cases.

Similar expressions for the current may be derived from equation (4.19).

**The Input Impedance of Resonant and Anti-Resonant Lines**

Consider the short-circuited line. From (4.17) the input impedance of this line is

\[
Z_s = Z_0 \frac{\sinh 2\alpha l + j \sin 2\beta l}{\cosh 2\alpha l + \cos 2\beta l}
\]

\[
= Z_0 \frac{\alpha l + j \sin \beta l \cos \beta l}{\alpha^2 l^2 + \cos^2 \beta l}
\]

for lines of low loss.

For short-circuited lines of resonant length \( n \frac{\lambda}{2} \), \( \sin \beta l = 0 \), \( \cos \beta l = 1 \), and the input impedance is

\[
Z_s = Z_0 \frac{\alpha n}{2} = Z_0 \frac{\alpha}{2} \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right)
\]

from (3.16),

\[
= \frac{R \alpha l}{4} \text{ for the air-spaced line} \quad \ldots \quad (5.3)
\]

For similar lines of anti-resonant length \( (2n + 1) \frac{\lambda}{4} \), on the other hand, \( \cos \beta l = 0 \) and

\[
Z_s = \frac{Z_0}{\alpha (2n + 1) \frac{\lambda}{4}} = \frac{8Z_0^2}{R(2n + 1)^2} \quad \ldots \quad (5.4)
\]

In each case the input impedance is purely resistive; in the resonant line case it is of low value and tends to zero with the line resistance per unit length \( R \), and in the anti-resonant line case it is of correspondingly high value.

These expressions may be written \( \frac{1}{4} R \alpha \frac{\lambda}{2} \) and \( \frac{1}{8} R \alpha \frac{\lambda}{4} \)

respectively, where \( n' \) is the number of quarter-wavelengths in the respective lines. In this form they apply equally to the idealized resonant and anti-resonant open-circuited lines.

On substituting for \( R \) from (3.10) for the air-spaced coaxial line the first of these expressions becomes

\[
Z_{s}^{\text{resonant}} = 1.19 \times 10^{4} \cdot \frac{1}{\sqrt{f}} \left( \frac{1}{a \sqrt{\sigma_{a}}} + \frac{1}{b \sqrt{\sigma_{b}}} \right) \text{ ohms.} \tag{5.5}
\]

where \( a \) and \( b \) are in metres, \( \sigma = 5.9 \times 10^{7} \) for copper and \( f \) is in cycles per second.

It will be seen that \( Z_{s} \) varies inversely as \( \sqrt{f} \). Its order of magnitude may be judged by taking the particular case of a half-wave all-copper line of dimensions \( b = 5 \times 10^{-3} \) metres, \( a = 5 \times 10^{-4} \) metres at a frequency of \( 10^{8} \) c.p.s. (\( \lambda = 0.3 \) metre). For this line the input impedance is 0.215 ohm.

The anti-resonant input impedance of the air-spaced coaxial line is given by

\[
Z_{s}^{\text{anti-resonant}} = 1.6 \cdot \frac{1}{n'} \left( \frac{1}{a \sqrt{\sigma_{a}}} + \frac{1}{b \sqrt{\sigma_{b}}} \right) \text{ ohms.} \tag{5.6}
\]

For a quarter-wavelength of the line detailed above this impedance has the value \( 1.77 \times 10^{5} \) ohms.

The following points may be noted. The anti-resonant input impedance

(a) increases as the square root of the frequency;
(b) decreases with increase in the number of quarter-wavelengths in the line;
(c) has maximum value, for conductors of the same material, for \( \frac{b}{a} = 9.2 \);
(d) increases as \( b \) and \( a \) increase, in the above ratio.

**Lines as Low-Loss Reactances**

Referring again to (5.2), for the short-circuited line, it is seen that for values of \( \beta l \) for which \( \cos^{2} \beta l \gg \alpha^{2} l^{2} \), that is, for values of \( l \) somewhat displaced from \( \left( \frac{2n+1}{4} \right) \), this equation becomes

\[
Z_{s} = Z_{o} \frac{\alpha l}{\cos^{2} \beta l} + jZ_{o} \tan \beta l. \tag{5.7}
\]

The input impedance is thus sensibly a pure inductive reactance for line lengths lying between \( \frac{\lambda}{4} \), \( \frac{\lambda}{2} \), \( \frac{3\lambda}{4} \), \&c., and a pure capacitive reactance for lengths between \( \frac{\lambda}{4} \) and \( \frac{\lambda}{2}, \frac{3\lambda}{4} \), \&c., excepting the lengths proximate to \( \frac{\lambda}{4}, \frac{3\lambda}{4}, \&c. \) The nature of the variation with line length is shown in Fig. 27 (a). In particular, for \( l = \frac{\lambda}{8} \)

\[
Z_{s} = \frac{1}{2}Z_{o} + jZ_{o} \tan \beta l.
\]

and for \( l = \frac{3\lambda}{8} \), \( Z_{s} = \frac{3}{4}R \lambda - jZ_{o} \).

For the idealized open-circuited line, on the other hand,

\[
Z_{s} = Z_{o} \frac{\alpha l}{\sin^{2} \beta l} - jZ_{o} \cot \beta l. \tag{5.8}
\]

except for values of \( l \) close to \( \frac{\lambda}{2} \). The reactance variation with line length is shown in this case by Fig. 27 (b). For \( l = \frac{\lambda}{8} \), \( Z_{s} \) is here \( \frac{1}{4}R \lambda - jZ_{o} \), which represents a capacitive reactance \( Z_{o} \) with only one-third the loss resistance of the \( \frac{3\lambda}{8} \) short-circuited line. For reasons already discussed, it will be appreciated, however, that the relation
between input reactance and line length expressed by (5.8) will not be adhered to precisely for practical open-ended lines.

Fig. 27. — Input reactance variation for short- and open-circuited lines.

THE TAPPED TRANSMISSION LINE

Consider a transmission line of length \( l \) shorted at both ends and tapped at an intermediate point as shown in

Fig. 28. — Tapped short-circuited line.

It follows from (5.1) that the admittance at the tapping point is given by

\[
Y_t = G_t + jB_t = \frac{1}{Z_0} \{ \coth \beta l_1 + \coth \beta l_2 \}
\]

in which

\[
G_t = \frac{1}{Z_0} \left\{ \frac{\sinh 2\alpha l_1}{2(\sinh^2 \alpha l_1 + \sin^2 \beta l_1)} + \frac{\sinh 2\alpha l_2}{2(\sinh^2 \alpha l_2 + \sin^2 \beta l_2)} \right\}
\]

\[
B_t = -\frac{1}{Z_0} \left\{ \frac{\cot \beta l_1}{1 + \frac{\sin^2 \alpha l_1}{\sin^2 \beta l_1}} + \frac{\cot \beta l_2}{1 + \frac{\sin^2 \alpha l_2}{\sin^2 \beta l_2}} \right\}
\]

If, now, the lengths \( l_1 \) and \( l_2 \) are such that

\[
\sinh^2 \alpha l_1 < \sin^2 \beta l_1 \quad \text{and} \quad \sinh^2 \alpha l_2 < \sin^2 \beta l_2,
\]

these expressions reduce to the forms

\[
G_t = \frac{1}{Z_0} \left\{ \frac{\alpha l_1}{\sin^2 \beta l_1} + \frac{\alpha l_2}{\sin^2 \beta l_2} \right\}
\]

\[
B_t = -\frac{1}{Z_0} \{ \cot \beta l_1 + \cot \beta l_2 \}
\]

Two cases are of particular interest:

(a) \( l = \frac{\lambda}{2} \)

Here \( \beta(l_1 + l_2) = \pi \), \( \cot \beta l_1 = -\cot \beta l_2 \) and

\[
\sin^2 \beta l_1 = \sin^2 \beta l_2,
\]

so that subject to the above-mentioned conditions,

\[
G_t = \frac{1}{Z_0} \left\{ \frac{\alpha(l_1 + l_2)}{\sin^2 \beta l_1} \right\} ; \quad B_t = 0
\]

Looked at from the tapping point the half-wave line thus behaves as a pure resistance of value

\[
R_t = Z_0 \frac{\sin^2 \beta l_1}{\alpha l} = \frac{4Z_0^2}{\alpha l} \sin^2 \frac{2\pi l_1}{\lambda}
\]

(b) \( l = \frac{\lambda}{4} \)
Here \( \beta(l_1 + l_2) = \frac{\pi}{2} \), \( \cot \beta l_3 = \tan \beta l_1 \), \( \sin^2 \beta l_2 = \cos^2 \beta l_1 \), so that
\[
G_t = \frac{1}{Z_0} \left[ \frac{\pi}{\sin^2 \beta l_1} + \frac{\pi}{\cos^2 \beta l_1} \right] ; \quad B_t = -\frac{2}{Z_0 \sin 2\beta l_1}
\]

\( G_t \) will be small for all significant positions of the tapping point and the system behaves at this point as an inductive reactance of value
\[
X_t = \frac{1}{2} Z_0 \sin \frac{2\pi l_1}{\lambda}
\]
shunted by a high resistance.

Similar consideration of the quarter-wave line open-circuited (ideally) at both ends will show that this system behaves at the tapping point as a capacitive reactance of value given by the same expression. The \( \lambda/4 \) line open at one end and closed at the other, on the other hand, provides a sensibly pure resistance at the tapping equal to
\[
\frac{8Z_0^2 \sin^2 2\pi \frac{l_1}{\lambda}}{RL} \sin 2 \pi \frac{l_1}{\lambda}
\]

As mentioned in the first chapter, the latter system has been employed for the frequency stabilization of valve oscillators in the same manner as quartz crystals at lower frequencies. The general requirements of a circuit to be used for this purpose are that its conductance \( G_t \) at the frequency to be stabilized must be as low as possible and the incremental susceptance introduced by a small departure from this frequency as high as possible. The characteristics of the tapped quarter-wave line, open at one end and closed at the other as shown in Fig. 8, from this point of view have been investigated both mathematically and experimentally by W. E. Willshaw,* who makes a final point which is of general significance. It is that the minimum length of connecting lead between the tapping point and the external circuit to which it is connected may become an appreciable fraction of a wavelength at very high frequencies. In consequence, unless special precautions are taken the impedance existing across the connecting leads remote from the tapping point may be greatly different from an anticipated theoretical value.

**THE Q VALUE OF RESONANT LINES**

The \( Q \) value of a resonant circuit of concentrated inductance, capacitance and resistance is defined as \( \frac{L_0 \omega}{R} \)
where \( \omega \) is the resonant angular frequency \( \frac{1}{\sqrt{LC}} \). This ratio may be written in more general form as follows:

\[
Q = \frac{L_0}{R} = 2\pi f \frac{\frac{1}{2} L I^2}{R} \quad \text{where } I \text{ is the current amplitude at resonance,}
\]

\[
= \frac{2\pi f}{RL} \text{ Stored energy in the magnetic field}
\]
\[
= \frac{2\pi f}{RL} \text{ Energy loss per cycle}
\]

Consider now a closed coaxial resonator as shown in Fig. 25, which will be supposed completely filled with a dielectric of relative permittivity \( \varepsilon \) and power factor \( \delta \). This will resonate at a free-space wavelength \( \lambda_0 \) for a length
\[
I = l = \frac{\lambda_0}{2} \sqrt{\frac{\varepsilon_0 \lambda_0}{e || 2} \quad \text{the current and voltage distributions being of the sinusoidal form shown in Fig. 25 (a). (They will be strictly sinusoidal only if the conductor and end-plate resistances are zero, but the discrepancy in practical resonators will be insignificant.)}
\]

If \( I \) is the current amplitude at the ends of the resonator, then the voltage amplitude half-way along its length is \( V = Z_0 I \), and if \( L, C, R \) and \( G \) are the line constants per unit length and \( R \), the resistance of each end-plate, we have

---

* W. E. Willshaw, Phil. Mag., 1940, 29, p. 572. See also M. Reed, Wireless Engineer, 1938, 15, p. 414.
Stored energy in the magnetic field

\[ \frac{1}{2} L \int_0^{\lambda/2} \left( I \cos \frac{2\pi}{\lambda} x \right)^2 \, dx = \frac{1}{\lambda} L \beta I^2 \]

Rate of energy loss in the line conductors, end plates and dielectric

\[ -\frac{1}{2} \left( R \int_0^{\lambda/2} \left( I \cos \frac{2\pi}{\lambda} x \right)^2 \, dx + 2 R_e I^2 + G \int_0^{\lambda/2} \left( I \sin \frac{2\pi}{\lambda} x \right)^2 \, dx \right) = \frac{1}{\lambda} \left( R \lambda + 2 R_e + G Z_0^2 \right) I^2 \]

The Q value of the system is thus

\[ Q = \frac{\frac{1}{R} + \frac{8}{L} \frac{R_e}{\lambda} + \frac{G Z_0^2}{L} \frac{1}{\lambda}}{\frac{1}{L} + \frac{1}{Z_0^2} \tan \delta} \]  \hspace{1cm} (5.9)

But \( G = C \omega \tan \delta \), so that \( \frac{G Z_0^2}{L} = \frac{C}{L} \frac{Z_0^2 \tan \delta}{\tan \delta} \),

\[ \frac{R}{L \omega} = \frac{1}{2} \sqrt{\pi \mu_0 \sigma} \frac{\log_{10} \frac{b}{a}}{\log_{10} \frac{b}{a}} \] from (3.10), the resonator being supposed constructed from the same metal throughout, and

\[ \frac{R_e}{L \omega} = \frac{1}{2} \sqrt{\pi \mu_0 \sigma} \] using (4.10), the depth of the equivalent current sheet being as given by \( \frac{t}{n} \), equation (3.7).

The expression for the Q value of the system is thus

\[ Q = \frac{1}{\frac{1}{R} + \frac{8}{L} \frac{R_e}{\lambda} + \frac{G Z_0^2}{L} \frac{1}{\lambda}} \] \hspace{1cm} (5.10)

\[ \frac{1}{\sqrt{\pi \mu_0 \sigma} \left( \frac{1}{2} \log_{10} \frac{b}{a} + b \frac{4}{\lambda} \right)} + \tan \delta \]

For the air-filled resonator, \( \tan \delta = 0 \), and, if the loss in the ends be ignored, this reduces to

\[ Q = \frac{L \omega}{R} = \sqrt{\pi \mu_0 \sigma} \left( \frac{2 \log_{10} \frac{b}{a}}{1 + \frac{1}{b}} \right) \]

\[ = 9.2 \times 10^{-5} \sqrt{\sigma} \frac{\log_{10} \frac{b}{a}}{1 + \frac{1}{b}} \]  \hspace{1cm} (5.11)

from which the following points may be noted:

1. \( Q \) increases as the square root of the frequency,
2. it attains maximum value for conductors of the same material for \( \frac{b}{a} = 3.6 \)—the condition for minimum (conductor) attenuation, p. 51,
3. it increases as \( b \) and \( a \) increase, in this ratio;
4. it is independent of the number of quarter wavelengths in a resonator, in which respect it differs from the input impedance of resonant lines;
5. it is related to the input impedance of the antiresonant line by the expression

\[ Z_n = \frac{4 Z_0^2 Q}{\pi n} \]

where \( n' \) is the number of quarter-wavelengths in the line.

For a copper line of dimensions

\[ a = 5 \times 10^{-4} \text{metres}, \quad b = 5 \times 10^{-3} \text{metres}, \]

the Q value at 10⁹ c.p.s. is about 1,000. This figure would be increased considerably, of course, by increasing \( b \) and using the optimum \( \frac{b}{a} \) ratio, but even so it is much higher than could possibly be obtained in a circuit of concentrated inductance and capacitance elements.
MEASUREMENT OF THE RELATIVE PERMITTIVITY AND POWER FACTOR OF DIELECTRIC MATERIALS BY USE OF A CLOSED COAXIAL RESONATOR

Fig. 29 illustrates a closed coaxial line of length \( l \) containing at its centre a disk of solid dielectric of thickness \( d \) and relative permittivity \( \frac{\varepsilon_1}{\varepsilon_0} \), having its faces in the transverse plane. The introduction of the disk will clearly increase the resonant wavelength relative to the value for a wholly air dielectric, and measurement of this change of resonant wavelength affords a means of determining the permittivity of the material. For this purpose it is permissible to neglect the resistance losses in the conductors and dielectric loss in the disk since their effect on the field distribution and on the resonant wavelength is very small for lines of practical significance.

Reckoning \( x \) in this case from the centre line of the system the general equations giving the voltage and current distributions in the air and solid dielectric portions are:

\[
\begin{align*}
\frac{x}{l} &> \frac{d}{l} \quad V = A e^{-j\beta x} + B e^{j\beta x} \quad \ldots \ldots (a) \\
I & = \frac{1}{Z_0} \{ A e^{-j\beta x} - B e^{j\beta x} \} \quad \ldots \ldots (b) \\
\frac{d}{l} < \frac{x}{l} &< \frac{d}{l} \quad V = A_1 e^{-j\beta x} + B_1 e^{j\beta x} \quad \ldots \ldots (c) \\
I & = \frac{1}{Z_{01}} \{ A_1 e^{-j\beta x} - B_1 e^{j\beta x} \} \quad \ldots \ldots (d)
\end{align*}
\]

in which \( \beta, \beta_1 \) are the phase constants and \( Z_0, Z_{01} \) the characteristic impedances of the air and solid dielectric regions respectively.

The justification for this representation in terms of voltages and currents, as against a more fundamental one in terms of the electric and magnetic fields, is that, the surfaces of the solid dielectric disk and of the resonator end-plates being supposed transverse to the conductor direction, the boundary conditions which require to be satisfied at these surfaces can be satisfied (see page 64) by the purely transverse field components on which the validity of the voltage and current equations rests.

The boundary conditions to be applied to these equations are that \( V = 0 \) at the short-circuiting plates at \( x = \pm \frac{l}{2} \), and that \( V \) and \( I \) must be continuous at the surface of the dielectric disk at \( x = \pm \frac{d}{2} \). When, furthermore, the system is in resonance there will be a current node at \( x = 0 \) so that

\( o = A_1 - B_1 \) from (d)

These boundary conditions lead to the following relations:

\[
\begin{align*}
o & = A e^{-j\beta_1 \frac{l}{2}} + B e^{j\beta_1 \frac{l}{2}} \\
2 A_1 \cos \beta_1 \frac{d}{2} & = A e^{-j\beta \frac{d}{2}} + B e^{j\beta \frac{d}{2}} \\
-2 j A_1 \sin \beta_1 \frac{d}{2} & = \frac{Z_{01}}{Z_0} \{ A e^{-j\beta \frac{d}{2}} - B e^{j\beta \frac{d}{2}} \}
\end{align*}
\]
The second and third of these equations give
\[ A = 2A_1 e^{j\frac{d}{2}} \left( \cos \beta \frac{d}{2} - \frac{Z_0}{Z_{e1}} \sin \beta \frac{d}{2} \right) \]
\[ B = 2A_1 e^{-j\frac{d}{2}} \left( \cos \beta \frac{d}{2} + \frac{Z_0}{Z_{e1}} \sin \beta \frac{d}{2} \right) \]

On substituting in the first equation and rearranging, the condition for resonance follows as
\[ \cot \beta \frac{d}{2} \cot \beta \frac{l - d}{2} = \frac{Z_0}{Z_{e1}} \]
or, expressed in terms of the relative permittivity of the disk and the free space wavelength for resonance,
\[ \cot \pi \frac{d}{\lambda_0} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} \cot \pi \frac{l - d}{\lambda_0} = \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} \]

This is a particular case of the general result obtained by H.R.L. Lamont * for a dielectric specimen placed in any position in the resonator. It enables the relative permittivity to be determined from a measurement of the resonant wavelength.

For reasonably thin specimens of low relative permittivity this relation may be used in the more convenient approximate form
\[ \frac{\varepsilon_1}{\varepsilon_0} = \frac{\pi \frac{l - d}{\lambda_0}}{\pi \frac{d}{\lambda_0} \left( 1 + \frac{3}{\pi} \cot \pi \frac{l - d}{\lambda_0} \right)} \]

The error involved in this approximation is less than \( \frac{1}{2} \) per. cent. for values of \( d/l \) less than 0.3.

The voltage and current distributions at resonance may be obtained in terms of \( A_1 \) on substitution for \( A \) and \( B \) in equations (5.12). Given these, an expression for the \( Q \) value of the system, taking account both of conductor resistance and the power factor of the disk material, can be derived in the manner described in the preceding section. In consequence of the latter source of loss this \( Q \) value will be lower than that of the resonator when completely air-filled, and a measurement of the change of \( Q \) which results from insertion of the specimen affords a means of determining the power factor of the material.

**The Measurement of \( Q \) Value**

It is easily shown for the resonant \( L, C, R \) circuit that the \( Q \) value \( \frac{L_0}{R} \) is equal to \( \frac{\omega}{\Delta \omega} \) or \( \frac{f}{\Delta f} \) where \( \Delta f \) is the width of the resonance curve in frequency at \( \frac{1}{\sqrt{2}} \) of its height at the resonant frequency \( f \). The same relation applies for the resonator. The \( Q \) value may be derived, therefore, by delineating the resonance curve by incremental frequency change about the resonant value. The voltage injection may be carried out by means of a small loop penetrating an end-plate of the resonator in the manner shown in Fig. 25, and the current response recorded on a crystal or thermocouple connected in circuit with a second, similarly situated, loop.

In view of the high \( Q \) values obtainable the required frequency change is small and adequate accuracy of measurement may not be obtained by determining this from the difference of two wavemeter readings. The \( Q \) value of the wavemeter may well be lower than that of the resonator under investigation. It is preferable to measure the frequency values direct by reference to a quartz crystal. The extension of heterodyne methods to the region of very high frequencies has been discussed, in as yet unpublished reports, by Dr. L. Essen.*

In general, the \( Q \) value obtained experimentally will be less than the theoretical figure, and, among the factors likely to cause this, effects associated with the presence of the energizing and measuring loops are likely to be the

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* Radio Department, National Physical Laboratory.
most important. If reliable measurements are to be obtained, absence of reaction between the resonator and the source of oscillations must be ensured, and for this purpose it is desirable to introduce a loss of the order of 10 db. in the feeding line. Furthermore, the response of the detector system should be made as sensitive, and frequency unselective, as possible.

Alternatively, the Q value of a line resonator may be determined by incremental change of line length about the resonant value, though in so far as this method involves the use of a sliding piston at one end of the resonator there is always the risk of contact difficulties.

**RESONANT LENGTHS OF A LINE TERMINATED IN ARBITRARY IMPEDANCES AT BOTH ENDS**

Fig. 30 represents a line of length l and constants $Z_0$, $a$ and $\beta$ terminated at its two ends in impedances $Z_1 = R_1 + jX_1$, $Z_2 = R_2 + jX_2$. In order to keep the system

![Fig. 30](image)

Fig. 30.

general the voltage injecting loop will be supposed located at an arbitrary distance $l_1$ from $Z_1$—point C—and we shall consider the current $I$ which an E.M.F., $E$, injected in series with the line at this point produces at a point distant $l_2$ from $Z_2$. A current measuring loop will be supposed loosely coupled to the line at this point, $D$.

The most convenient means of calculating $I$ is by use

of Thevenin's theorem * which states, taking the elementary system of Fig. 31 as example, that the current flow in the inter-connexion between the parts $A$ and $B$ of that system

![Fig. 31](image)

Fig. 31.

is given by $\frac{V_{oc}}{Z_A + Z_B}$, where $V_{oc}$ is the voltage existing across the interconnexion when the circuit is broken at this point, and $Z_A$ and $Z_B$ are the impedances of $A$ and $B$ reckoned from the disconnexion.

Thus let us calculate the voltage developed across the line at $D$ when, as shown in Fig. 32 (a), it is supposed

![Fig. 32](image)

Fig. 32.

severed at this point. In the equivalent circuit of Fig. 32 (b), $Z^1$ and $Z^{11}$ are the input impedances of lines of lengths $l_1$ and $l_3$ terminated respectively in $Z_1$ and an open circuit. Thus from (4.17)

* This procedure has been adopted, in a similar connexion, in an unpublished report by the late D. H. T. Gant.
constant throughout, the variation of \( l \) then showing itself only in \( l_s \). This requires that the injecting loop shall remain fixed in position relative to the \( Z_1 \) end, and the measuring loop likewise relative to the \( Z_3 \) end, of the line. In this event the numerator of (5.13) is unchanged throughout a variation of \( l \) and this equation may be written

\[
I = \frac{\text{constant}}{\sinh (Pl + \psi_1 + \psi_3)}
\]

of which the square of the modulus is given by

\[
|I|^2 \propto \frac{1}{\sinh^2 (P(1 + l_s + l_1)) + \sin^2 (\beta l + \psi_1 + \psi_3)} \quad (5.14)
\]

If the line attenuation \( \alpha l \) is negligibly small compared with \( u_1 + u_4 \), the current passes through maximum values as \( l \) is varied for line lengths \( l \), given by

\[
\beta l_4 + \psi_1 + \psi_3 = n\pi 
\]

where \( n \) is a positive integer or zero.

**THE ‘CHIPMAN’ METHOD OF IMPEDANCE MEASUREMENT** *

The above analysis leads to a method of impedance measurement first formulated by Chipman. Thus, suppose that it is required to determine the value of the impedance \( Z_1 \). This follows on evaluation of \( u_1 \) and \( v_1 \).

\( v_1 \) may be derived from (5.15) as follows:

1. Replace the unknown impedance by a short-circuiting plate for which both \( u \) and \( v \) are zero, and measure a resonant line length \( l_{r(s.c.)} \).

2. Attach the unknown impedance \( Z_1 \) and measure the corresponding resonant line length \( l_{r(1)} \).

Then from (5.15)

\[
v_1 = \frac{2\pi}{l} (l_{r(1)} - l_{r(s.c.)}) 
\]

\( \frac{l}{l_{r(s.c.)}} \) being unchanged.

\( u_1 \) is derived from measurement of the width of the

resonance curve of the system—delineated by change of line length—first with the line short-circuited and then with the short replaced by the unknown impedance.

The form of the resonance curve is

\[
\left| \frac{I}{I_r} \right| = \left\{ \frac{\sinh^2 (al_r + u_1 + u_2)}{\sinh^2 (al + u_1 + u_2) + \sin^2 (bl + v_1 + v_2)} \right\}^{1/2},
\]

from which it follows that \( I \) falls to \( \frac{1}{\sqrt{2}} I_r \) for changes of \( l \) on the two sides of the resonant length \( \delta l' \) and \( \delta l'' \) respectively, given by the relations

\[
\sinh^2 \{x(l_r + \delta l') + u_1 + u_2\} + \sin^2 \beta \delta l' = 2 \sinh^2 (x l_r + u_1 + u_2)
\]

\[
\sinh^2 \{x(l_r - \delta l'') + u_1 + u_2\} + \sin^2 \beta \delta l'' = 2 \sinh^2 (x l_r + u_1 + u_2)
\]

If, now, \( \delta l' \) and \( \delta l'' \) are small compared with \( l_r \), it follows that the resonance curve is symmetrical and that

\[
\sinh (x l_r + u_1 + u_2) = \sin \frac{\beta}{2} \Delta l, \quad \text{where } \Delta l \text{ is the width } (\delta l' + \delta l'') \text{ of the resonance curve at } \frac{1}{\sqrt{2}} \text{ of its height.}
\]

Thus if

1. \( \Delta l_{(c)} \) is the width of the resonance curve when the line is closed in a short-circuited plate

\[
\sinh (x l_{(c)} + u_2) = \sin \frac{\beta}{2} \Delta l_{(c)} . \quad \text{(5.18)}
\]

2. \( \Delta l_i \) is the width with the unknown impedance attached

\[
\sinh (x l_i + u_1 + u_2) = \sin \frac{\beta}{2} \Delta l_i . \quad \text{(5.19)}
\]

from which

\[
u_1 = \sinh^{-1} \sin \frac{\pi}{\lambda} \Delta l_1 = \sinh^{-1} \left\{ \sin \frac{\pi}{\lambda} \Delta l_{(c)} \right\}
\]

\[
- \frac{\alpha (l_{(c)} - l_i)}{2} \Delta l_i . \quad \text{(5.20)}
\]

Given \( u_1 \) and \( v_1 \) known, \( Z_1 \) follows as

\[
Z_1 = R_1 + jX_1 = Z_0 \tanh (u_1 + jv_1)
\]

These derivations require that the constants \( Z_0 \) and \( \alpha \) of the measuring line must be known.

Where the former cannot be calculated reliably it may be determined experimentally by using the line to measure the input impedance of a second line of calculable characteristic impedance and by utilizing the circle diagram technique described later on page 142.

The attenuation constant \( \alpha \) may be determined from measurements of the width of the resonance curve with the measuring line short-circuited (or open-circuited) at a succession of resonant line lengths. Thus if \( \Delta l' \) and \( \Delta l'' \) are the widths at \( \frac{1}{\sqrt{2}} \) height of the resonance curves obtained at adjacent resonant length settings

\[
\frac{\sin \frac{\beta}{2} \Delta l'}{\sinh (\alpha l_r + u_2)} = \frac{\sin \frac{\beta}{2} \Delta l''}{\sinh \left( \alpha \left( l_r + \frac{\lambda}{2} \right) + u_2 \right)} = 1 . \quad \text{(5.21)}
\]

from (5.17). For accuracy of impedance measurement \( u_i \) should be kept small; it may, of course, be made zero by closing the \( Z_1 \) end of the line in a short-circuiting plate. In this case the denominators in (5.21) may be replaced by the corresponding angles with the result that

\[
\alpha = \frac{\sin \frac{\pi}{\lambda} \Delta l'' - \sin \frac{\pi}{\lambda} \Delta l'}{\frac{\lambda}{2}} .
\]

Relations analogous to (5.16) and (5.20) could have been obtained in terms of the reflection coefficients of the \( Z_1 \) and \( Z_4 \) terminations. This is the procedure adopted by Chipman, and more recently by Essen, from whose publications the appropriate relations and a description of measuring

* I.E.E. Journal, 1944, 91, Part III, p. 84.
apparatus may be obtained. The latter has given a very comprehensive treatment of this method of impedance measurement, and his paper should be read at this stage.

In each case the current response of the system is obtained not by use of a pick-up loop in the manner illustrated in Fig. 30, but on a thermo-junction connected to the line in the position occupied by the impedance $Z_s$ in that figure. The length $l_s$, being in this case zero, the condition specified on page 109 is satisfied. The line length is changed by sliding the thermo-junction, together with short-circuiting bridges suitably situated behind it, along the line conductors.

A method of impedance measurement similar in principle to the one discussed above has been described by Kaufmann.* His apparatus consisted of an open wire twin line shorted at both ends, one short-circuiting link being fixed and the other capable of movement along the line for the purpose of resonance adjustment. The impedance to be measured was joined across the line at an intermediate point, chosen by experiment to suit the order of magnitude of this impedance. In a manner analogous to the above the reactive and resistive components of the impedance were determined from the change of resonant line length and width of resonance curve resulting from its attachment.

A further method, employing a different principle, which has advantages where high impedances are to be measured at the lower end of the V.H.F. range (say, 50-100 Mc/sec.) has been described by Miller and Salzberg.† The principle of the method is illustrated by Fig. 33, which shows a line of length less than a quarter-wavelength, short-circuited at one end and tuned by a variable condenser at the other. A voltmeter is connected across the condenser and a constant E.M.F. induced in the line.

The first step is to connect the impedance to be measured across $C$ and to adjust the latter to give resonance. The equivalent shunt reactance of the impedance then follows from the difference between this value of $C$ and the one required for resonance with the impedance unconnected.

With the system now in the latter condition the second step is to slide a known 'non-inductive' resistance $R$ along the line until a position is found for which the voltmeter reading is the same as it was with the impedance connected across $C$. This enables the equivalent shunt resistance to be calculated as follows:

\[ V_y = V_s \left( \frac{\sin Py}{\sin Pl} \right) \]
\[ = V_s \left( \frac{\sin \frac{2\pi}{\lambda} y}{\sin \frac{2\pi}{\lambda} l} \right) \text{ neglecting line losses.} \]

On the assumption that the connexion of a resistance $R$ across the line at a distance $y$ from the short-circuited end does not disturb the voltage distribution significantly, the power loss in this resistance is

\[ \frac{V_s^2}{R} \left\{ \frac{\sin \frac{2\pi}{\lambda} y}{\sin \frac{2\pi}{\lambda} l} \right\}^2 \]
The resistance which, connected across \( C \), gives the same power loss, is

\[
R \left\{ \frac{\sin \frac{2\pi l}{\lambda}}{\sin \frac{2\pi l}{\lambda} \cdot y} \right\}^2
\]

the desired equivalent shunt resistance referred to above.

An approach to a non-inductive resistance at moderate frequencies may be obtained in convenient form for attachment to an open wire twin line by use of rod type carbon resistors. There is good evidence to show that the resistance of these units at high frequencies does not differ appreciably from the d.c. resistance up to values of the order of 1,000 ohms.*

The method can be, and has been, employed for the measurement of low impedances by connecting them across the line at a point between the tuning condenser and the short circuit. For further information reference should be made to the original paper of Miller and Salzberg.

**MEASUREMENT OF THE CHARACTERISTIC CONSTANTS OF TRANSMISSION LINES (e.g. LENGTHS OF FLEXIBLE FEEDER)**

It follows from equation (4.17) that the input impedances of short-circuited and open-circuited lines of length \( l \) are given respectively by

\[
Z_{\text{sc}}(\text{a.c.}) = \frac{Z_0 \tanh P l = |Z_{\text{a.c.}}|}{\alpha_{\text{a.c.}}} \quad (5.22)
\]

\[
Z_{\text{oc}}(\text{a.c.}) = \frac{Z_0 \coth P l = |Z_{\text{a.c.}}|}{\beta_{\text{a.c.}}} \quad (5.23)
\]

If, therefore, measurements of these two impedances are made on a suitable length of feeder, using either the Chipman or Brückmann technique, its \( Z_0 \) and \( P \) values may be derived from the relations

\[
Z_0 = \sqrt{\frac{|Z_{\text{a.c.}}|}{|Z_{\text{a.c.}}|}} \cdot \frac{1}{2}(\alpha_{\text{a.c.}} + \beta_{\text{a.c.}}) \quad (5.24)
\]

\[
\tanh P l = \tanh (\alpha + j\beta) l = \sqrt{\frac{|Z_{\text{a.c.}}|}{|Z_{\text{a.c.}}|}} \cdot \frac{1}{2}(\alpha_{\text{a.c.}} - \beta_{\text{a.c.}}) \quad (5.25)
\]

If the latter is expressed \( \tanh P l = L + jM \), then \( \alpha \) and \( \beta \) may be derived from the relations

\[
\tanh 2\alpha l = 2L/(1 + L^2 + M^2) \quad (5.26)
\]

\[
\tanh 2\beta l = 2M/(1 - L^2 - M^2) \quad (5.27)
\]

These equations hold whatever the values of \( Z_{\text{a.c.}} \) and \( Z_{\text{a.c.}} \) but, as discussed by Essen, there are practical reasons for making them approximately equal in magnitude and phase angle, though with the latter of opposite sign in the two cases, by suitable adjustment of feeder length or of frequency. For this condition the length is \( (2n + 1)\frac{\lambda}{4} \), where \( \lambda \) is the wavelength in the feeder.

The drawback to relying on two values of input impedance only for the derivation of \( Z_{\text{a.c.}} \alpha \) and \( \beta \) is that a small spurious and unknown reactance, external to the feeder under test, is likely to exist at the point of its connexion to the measuring line. The inaccuracy for which this will be responsible is reduced to a minimum by the choice of feeder length mentioned above, but it is nevertheless desirable to measure \( Z_{\text{a.c.}} \) and \( Z_{\text{a.c.}} \) for two lengths differing by \( \frac{\lambda}{4} \) (or at two frequencies such that the electrical length of the test feeder differs by \( \frac{\lambda}{4} \)).

Alternatively, and preferably where time permits, by determining the input impedance in either the short- or open-circuited condition for a range of lengths differing overall by one half-wavelength, and by utilizing the circle diagram technique discussed in the next chapter,* the effect of the junction impedance can be entirely eliminated.

For air-spaced lines the wavelength in the line is the

* See page 143.
free-space value \( \lambda_0 \) so that the phase constant \( \beta = \frac{2\pi}{\lambda} \) may be calculated from a wavemeter measurement. In solid or semi-solid core feeders the wavelength will be \( \sqrt{\frac{\varepsilon_0}{\varepsilon}} \lambda_0 \), where \( \varepsilon \) is the relative permittivity of the feeder dielectric. Since this may not be known reliably, the wavelength in the feeder is preferably measured directly.

It has been shown in Fig. 27 that the input reactance of a line passes through zero value for line lengths differing by \( \frac{\lambda}{2} \). The determination of \( \lambda \) only requires, therefore, that a feeder length shall be found for which the input reactance is zero, and that the length shall then be changed until this condition is restored. This cutting of the feeder is performed most conveniently when the free end is left open-circuited. It may prove difficult to cut off the exact length to give the required resonant setting, but since the reactance variation with length is sensibly linear in the region of zero reactance, linear extrapolation may be used to obtain the required value of line length.

Fuller information on the above points is given in the paper by Essen referred to previously.

**Composite Lines as Wave Filters**

At the higher radio frequencies it becomes increasingly difficult to construct coil and condenser filter networks to give a predetermined performance on account of the small sizes of the circuit elements required and the large effects of the interconnecting wires. The former difficulty can be removed by the use of short lengths of transmission lines as the circuit elements, but the latter remains, though it takes a somewhat different form. For the reasons discussed on pages 62–9, the process of interconnecting short lengths of line involves inevitably disturbance of the purely transverse field distribution on which the transmission line equations rest, so that expressions such as (5.3), (5.4), (5.7), (5.8), will not be expected to describe more than approximately the behaviour of the elements concerned when they are built into a composite structure. These equations and expressions afford, nevertheless, a reliable basis for preliminary design.

**Fig. 34.—Element of a line filter.**

Fig. 34 illustrates one section of a line filter consisting of a length \( 2l_1 \) of line of characteristic impedance \( Z_1 \) shunted at its centre by a short-circuited length \( l_2 \) of line of characteristic impedance \( Z_2 \). It has been shown by Mason and Sykes * that the behaviour of an infinite succession of such sections can be expressed in terms of a propagation constant per section, \( \gamma \), and a characteristic impedance, \( Z_0 \), in a manner analogous to the representation of normal coil-condenser filter networks.

For the system of Fig. 34, it is shown that these quantities are given by the expressions

$$\cosh \gamma = \cos 2\beta l + \frac{Z_1}{2Z_2} \sin 2\beta l \cot \beta l$$

$$Z_0 = Z_1 \sqrt{\frac{1 + \frac{Z_1}{2Z_2} \tan \beta l \cot \beta l}{1 - \frac{Z_1}{2Z_2} \cot \beta l \cot \beta l}}$$

line losses being neglected.

The cut-off characteristics of the system may be illustrated by taking the particular case in which $Z_1 = 2Z_2$.

In this case

$$\cosh \gamma = \cos 2\beta l + \sin 2\beta l \cot \beta l$$

on expressing $\gamma$ in the general form $A + jB$.

This relation requires that $\sinh A \cdot \cos B = \cos \gamma$, a condition which is satisfied by either $\sinh A = 0$ or $\sin B = 0$. The first gives $A = 0$, corresponding to unattenuated propagation through the system. The wavelength limits of this "pass-band" are then given by the supplementary requirement that

$$\cosh A \cdot \cos B = \cos \gamma = \frac{\sin (2\beta l + l_2)}{\sin \beta l_2}$$

Since the limiting values of $\cos B$ are $\pm 1$, the limiting wavelengths are given by

$$\sin \beta (2\beta l + l_2) = \pm \sin \beta l_2$$

namely, by $\lambda = (2n + 1)\frac{\lambda}{4}$ and $4l_1$.

Other cases are considered by Mason and Sykes. They also discuss the use of composite lines as wide-band transformers.

CHAPTER VI

IMPEDANCE TRANSFORMATION—THE USE OF THE CIRCLE DIAGRAM TECHNIQUE

The input impedance of a line of length $l$ terminated in an impedance $Z_\tau$ is given by the expression

$$Z_8 = Z_0 \cdot \frac{Z_\tau}{Z_0} + \tanh Pl \cdot \frac{Z_\tau}{Z_0} \cdot \frac{1}{1 + \frac{Z_\tau}{Z_0} \tanh Pl} \cdots \cdots (4.17)$$

which signifies that the line acts as an Impedance Transformer of complex ratio

$$Z_8 = \frac{Z_\tau}{Z_0} + \tanh Pl \cdot \frac{Z_\tau}{Z_0} \cdot \frac{1}{1 + \frac{Z_\tau}{Z_0} \tanh Pl}$$

The range of this ratio with variation of $l$ may be assessed by neglecting the line attenuation and replacing $\tanh Pl$ by $j \tan \beta l$. Thus $\tan \beta l$ varies between 0, corresponding to $l = \frac{\lambda}{2}$, when $\frac{Z_8}{Z_\tau} = 1$, and infinity, corresponding to $l = (2n + 1)\frac{\lambda}{4}$, when $\frac{Z_8}{Z_\tau} = \left(\frac{Z_0}{Z_\tau}\right)^3$.

A single or multiple half-wavelength line thus acts as a $1 : 1$ transformer, and the odd multiple quarter-wave line in such a way that $Z_8Z_\tau = Z_0^3$.

The quarter-wave transformer is the simplest form of matching device, and the method of its application was discussed in Chapter I. Since, however, the characteristic impedance of lines operated at very high frequencies is purely resistive, this device cannot by itself provide a match.
when the terminating impedance is reactive, and, as discussed in Chapter I, stub-matching methods are of much more general application.

The analytical solution of all matching problems rests on the utilization of equation (4.17). In many cases this can be performed algebraically without much difficulty by use of Kennelly’s tables and charts of hyperbolic functions, but for preliminary analyses it is often more convenient to make use of the graphical methods which have been developed, and in the case of complex systems of line elements these methods may afford the only practicable means of solution. Several forms of graphical treatment have been suggested,* but it is proposed to concentrate here on two forms of Circle Diagram † which have found wide application during recent years. The theory of these will be developed and their use illustrated by a number of simple examples.

**Theory of the Circle Diagrams of Impedance.** ‡ As a preliminary we shall ‘normalize’ the impedances $Z_\alpha$ and $Z_\gamma$, that is, express them in ratio to the characteristic impedance $Z_0$ of the line in question, and replace $P_l$ by a complex angle $u_0 + jv_0$, thus

$$Z_\alpha = \frac{Z_\alpha}{Z_0}; \quad Z_\gamma = \frac{Z_\gamma}{Z_0}$$

$$u_0 + jv_0 = P_l = \alpha l + \frac{2\pi}{A} l$$

(4.17) now becomes

$$Z_\alpha = \frac{Z_\gamma + \tan \theta (u_0 + jv_0)}{1 + Z_\gamma \tan \theta (u_0 + jv_0)}$$


‡ A number of alternative treatments of the theory of these diagrams in terms of reflection coefficients are given in the discussion to the above-mentioned paper by Jackson and Huxley.

when a further complex quantity $\psi = u + jv$ may be defined* such that

$$z_\alpha = \tanh \psi = \tanh (u_0 + jv_0),$$

to give

$$z_\alpha = \tanh \left\{ (u_0 + u_1) + j(v_0 + v_1) \right\}$$

The whole purpose of the circle diagram is to yield the quantities $u$ and $v$ which satisfy the relation

$$z = r + jx = \tanh (u + jv)$$

when $r$ and $x$ are given, or conversely to provide $r$ and $x$ when $u$ and $v$ are specified. The processes of addition and subtraction within the hyperbolic function may then be performed without difficulty.

**THE CARTESIAN GRID FORM OF CIRCLE DIAMON**

Consider the transformation

$$Z = r + jx = c \tanh \psi = c \tanh (u + jv) \quad (6.1)$$

in which the constant $c$ fixes the scale of $r$ and $x$.

First express the real quantities $u$ and $v$ as separate functions of $r$ and $x$.

Equation (6.1) gives

$$Z - c = \frac{e^{2(u + jv)} - 1}{e^{2(u + jv)} + 1}$$

or

$$Z - c = - e^{-2(u + jv)} \quad \ldots \quad (6.2)$$

Now, writing $\bar{Z} = r - jx$ as the complex conjugate of $Z$,

$$Z - c = \frac{r^2 + x^2 + c^2 - 2cr}{(Z + c)(Z - c)} = e^{-4u}$$

or

$$r^2 + x^2 + c^2 - 2cr \coth 2u = 0 \quad \ldots \quad (6.3)$$

Similarly

$$\frac{(Z - c)(\bar{Z} + c)}{(Z + c)(\bar{Z} - c)} = e^{-4u}$$

or

$$r^2 + x^2 + c^2 - 2cx \cot 2u = 0 \quad \ldots \quad (6.4)$$

Equations (6.3) and (6.4) may be written in the more convenient forms

$$(r - c \cot 2u)^2 + x^2 = c^2 \csc^2 2u \quad \ldots \quad (6.5)$$

$$r^2 + (x + c \cot 2u)^2 = c^2 \csc^2 2u \quad \ldots \quad (6.6)$$

* See page 87.
Reverting now to equation (6.1), the complex quantity \( z = r + jx \) is represented by a point \((r.x)\) in a plane Cartesian system of co-ordinates—the \( z \) plane, and \( \psi = u + jv \) by a point \((u.v)\) in a second plane—the \( \psi \) plane. These are shown in Fig. 35.

The transformation (6.1) associates points \((r.x)\) in the \( z \) plane with points \((u.v)\) in the \( \psi \) plane, and as we are not concerned with negative resistances, only the first and fourth quadrants in the \( z \) plane are significant, and \( u \) is always positive.

With each point \((u.v)\) in the \( \psi \) plane, one point \((r.x)\) only is associated, but with each \((r.x)\) point is associated an infinite sequence of points \((u.v \pm n\pi)\), where \( n \) is an integer. For convenience we shall consider only the range of \( v \) from 0 to \( \pi \).

Now let a point in the \( \psi \) plane move along a line \( u = \) constant from \( v = 0 \) to \( v = \pi \). The co-ordinates of the associated sequence of \( z \) points, as given by (6.6), must satisfy equation (6.5), and since \( u \) is a constant these points lie on a circle in the \( z \) plane, of centre \((c \cot 2u, 0)\) and radius \( c \coth 2u \), the circle being traversed once as \( v \) increases from 0 to \( \pi \). For each value of \( u \) there is a corresponding circle; these circles, \( u = \) constant, will be called ' \( u \) circles'. The point \( z = c + jo \) is the circle \( u = \) infinity, and the \( ox \) axis of the \( z \) plane the circle \( u = 0 \).

Analogously, if a point in the \( \psi \) plane traverses a line \( v = \) constant from \( u = + \) infinity to \( u = 0 \), the associated \( z \) points, as given by (6.5), must satisfy (6.6), and lie therefore on a circle of centre \((0, -c \cot 2v)\) and radius \( c \cosech 2v \). As \( u \) varies, however, only a part of the circle is traversed, since the \( z \) point corresponding to \( u = + \) infinity is \((c.o)\), for all values of \( v \), while that corresponding to \( u = 0 \) lies along the \( ox \) axis. The position of this point for a particular value of \( v \) follows on putting \( r = 0 \) in equation (6.6). These circular arcs, \( v = \) constant, may be called ' \( \psi \) arcs'. Alternatively, it is convenient to introduce a quantity \( n = \frac{v}{2\pi} = \frac{l}{\lambda} \), which has the significance of a length of line \( l \) divided by the wavelength \( \lambda \). Over the range of \( v \) from 0 to \( \pi \), \( n \) varies between 0 and 0.5. The circular arcs \( n = \) constant, represented, on replacing \( v \) by \( 2\pi n \) in (6.6), by

\[
\lambda^2 + (x + c \cot 4\pi n)^2 = c^2 \cosech^2 4\pi n. \quad (6.7)
\]

will be called ' \( n \) arcs'.

Since all the \( z \) points associated with the \( \psi \) points lying on the line \( u = u' \), say, will be on the circle \( u = u' \), and all the \( z \) points associated with the line \( v = v' \), \( n = n' \), say, will lie on the arc \( n = n' \), then the \( z \) point corresponding to \( \psi = u' + jv' \) must be the point of intersection of the \( u = u' \) circle and the \( n = n' \) arc. This is illustrated in Fig. 36.

\( A \) is the \( z \) point \( r' + jx' \) corresponding to the \( \psi \) point \( u' + jv' \).

\( B \) is the point \((c.0)\) common to all the \( \psi \) arcs.

Conversely, when \( z = r + jx \) is given, the corresponding values of \( u \) and \( v \) satisfying equation (6.1) are found from

\[
(z - c)^2 + (r - cx)^2 = c^2 \cosech^2 2u. \quad (6.8)
\]
the particular $u$ circle and $n$ arc which intersect at the point $(r, x)$. In order that it may be possible to obtain the $u, v$ values corresponding to any desired $z$, directly or by interpolation, the $z$ plane is covered with a mesh of $u$ circles

![Graph](image)

**Fig. 36.**

and $n$ arcs in the manner shown in Fig. 37. This mesh is constructed to the scale of $r$ and $x$ corresponding to $c = 1$; data for constructing a more complete form are given in Appendix I, p. 144, of this volume.

**THE 'POLAR' FORM OF CIRCLE DIAGRAM**

Consider again the transformation given by equation (6.1) in which the constant $c$ will now be taken as unity from the commencement. We shall seek to relate $z$ and $v$ through a third complex variable $t = p + jq$ as follows.

From (6.2) \[ e^{-2v} = \frac{z - 1}{z + 1} \]

In this relation put $t = e^{-2v}$; then $z + 1 = \frac{2}{t + 1}$. On
substituting \( p + jq, u + jv, \) and \( r + jx \) for \( t, \psi \) and \( z \) respectively these two equations become
\[
\begin{align*}
\rho + j\eta &= e^{-2\upsilon} (\cos 2\upsilon - j \sin 2\upsilon) \quad (6.8) \\
(r + 1) + jx &= \frac{2}{(\rho + 1) + jq} \quad (6.9)
\end{align*}
\]
thus linking through the point \((\rho, \eta)\) in the \( t \) plane corresponding points \((r, x)\) in the \( z \) and \( \psi \) planes.

The 'Polar' form of diagram is composed of two pairs of families of curves in the \( t \) plane; (1) the family pair, \( u = \) constant and \( v = \) constant (or writing \( n = \frac{v}{2\pi} \), as before, \( u = \) constant and \( n = \) constant); (2) the family pair, \( r = \) constant, \( x = \) constant.

(1) The \( u, n \) Families of Curves. It follows from (6.8) that
\[
\rho = e^{-2\upsilon} \cos 4\pi n; \quad \eta = -e^{-2\upsilon} \sin 4\pi n,
\]
whence
\[
\rho^2 + \eta^2 = e^{-4\upsilon} \quad \text{and} \quad \frac{\eta}{\rho} = -\tan 4\pi n \quad .\quad (6.10)
\]
Thus the curves \( u = \) constant are circles centred on the origin with radii \( e^{-2\upsilon} \), and the curves \( n = \) constant, a set of straight lines radiating from the origin with a slope \( -\tan 4\pi n \). A member of each family is shown in Fig. 38.

(2) The \( r, x \) Families of Curves. On rationalizing the right-hand side of (6.9) and rearranging, it follows that
\[
\left(\rho + 1 - \frac{1}{r + 1}\right)^2 + \eta^2 = \left(\frac{1}{r + 1}\right)^2
\]
\[
(\rho + 1)^2 + \left(q + \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2
\]
Thus the curves \( r = \) constant are circles of radii \( \frac{1}{r + 1} \), the centres of which lie at the points \( \left\{-1 + \frac{1}{r + 1}, 0\right\} \).

Any circle \( r = \) constant meets the \( p \) axis \( (q = 0) \) in the points \( \rho = -1 + \frac{1}{r + 1} \pm \frac{1}{r + 1} \), i.e. \( p = -1 \) and \( -1 + \frac{2}{r + 1} \).

All the \( r \) circles thus pass through the point \((-1, 0)\) of the \( t \) plane. This is illustrated in Fig. 39 (a).

The curves \( x = \) constant are circles of radii \( \frac{1}{x} \) having their centres at the points \( \left(-1, -\frac{1}{x}\right) \) on the ordinate through the point \((-1, 0)\), as shown in Fig. 39 (b). It is evident that the \( p \) axis is tangential to all the \( x = \) constant circles at this \((-1, 0)\) point. Since \( r \), the normalized resistance, is always positive, the portion of the \( t \) plane to the left of the ordinate through \((-1, 0)\) has no significance in the present application.

It may be shown that the square on the line of centres is equal to the sum of the squares on the radii, from which it follows that the \( r = \) constant, \( x = \) constant, circles form an orthogonal set.

The polar form of transmission line calculator devised
originally by P. H. Smith is shown in skeleton form in Fig. 40. This figure shows the \( r, \alpha \) and \( u \) families of circles and the radial \( n = \) constant lines superposed in the \( t \) plane, the \( n \) scale being given round the rim of the diagram. As here reproduced the real axis, \( p \), is vertical, and the imaginary axis, \( q \), horizontal, the origin being at the centre. In Smith's diagram the \( u \) circles and \( n \) lines are not shown directly, and use is made of a transparent arm, suitably graduated and provided with a slide, which is rotated about

*P. H. Smith, *Electronics*, 1939, 12, p. 29, and 1944, 19, p. 139.*
the centre point of the diagram. Data for constructing the Smith diagram are given in Appendix II, page 146, of this volume.

APPLICATION TO TRANSMISSION LINE PROBLEMS

For a transmission line of uniformly distributed constants $L$, $C$, $R$ and $G$ per unit length, operated at very high frequency, the attenuation coefficient $\alpha$ is given by the expression

$$\alpha = \frac{1}{2} \left( \frac{R}{Z_0} + GZ_0 \right)$$

nepers per unit length, the characteristic impedance $Z_0$ being a purely resistive quantity $\sqrt{L/C}$. Thus, as it appears in (6.1), $u_0 = \alpha l$, is expressed in nepers. It is general practice, however, to express the attenuation in decibels, and this has been taken account of in providing the data of Appendices I and II. In using the circle diagrams $u_0$ must, therefore, be converted to decibels by multiplying the value as calculated above by 8.686.

EVALUATION OF THE INPUT IMPEDANCE OF A LINE TERMINATED IN A KNOWN IMPEDANCE

(a) General. The terminating impedance must first be normalized to give

$$z_r = \frac{Z_r}{Z_0} = \frac{R_r + jX_r}{Z_0} = r_r + jx_r.$$

Then form a quantity $n_0 = \frac{l}{\lambda}$ where $l$ is the line length and $\lambda$ the wavelength in the line, and from knowledge of the line parameters $L$, $C$, $R$ and $G$, or otherwise, calculate

$$Pl = \alpha l + j2\pi n_0 = u_0 + j2\pi n_0,$$

$u_0$, as advised above, being expressed in decibels. If the dielectric of the line is other than air, it should be remembered that $\lambda = \sqrt{\frac{c}{e} \cdot \lambda_0}$, where $\epsilon/\epsilon_0$ is the relative permittivity of this dielectric and $\lambda_0$ is the free-space wavelength.

Consider first the Cartesian form of diagram. Locate on this diagram the point $(r_r, x_r)$, $P$, Fig. 41, and note the values $u_1$ and $n_1$ of the $u$ circle and $n$ arc intersecting at this point. Form $u = u_0 + u_1$ and $n = n_0 + n_1$ and locate the point $(r, x)$ representing the intersection of the $(u_0 + u_1)$ circle and the $(n_0 + n_1)$ arc. This point $Q$ is the representative point of the normalized input impedance $z_s$ of equation (4.17) since it satisfies (6.1) into which (4.17) was transformed. The actual input impedance is then obtained as $Z_0 z_s$.

In a large number of practical applications the line attenuation can be neglected, so that $u_0 = 0$. In this case the final representative point $z_s$ is located by running round the $u$ circle, $u = u_1$, from the arc $n_1$ to the arc $n_1 + n_0$. In using the Polar form of diagram, the point $(p, q)$ corresponding to the normalized impedance $r_r + jx_r$ is obtained.
as the point of intersection of the circles \( r_T = \text{constant}, \ x_T = \text{constant} \). The corresponding \( u \) and \( n \) values are then noted from the particular \( u \) circle \( (u = u_1) \) and \( n \) line \( (n = n_1) \) which intersect at this point \( (p,q) \). Finally, the desired input impedance is found by locating the point \( (u_0 + u_1, n_0 + n_1) \) on the \( u-n \) system, and observing the corresponding \( (r,x) \) values on the \( r-x \) system of curves.

![Diagram](image)

Fig. 42.

The procedure with the two forms of diagram being so similar, the succeeding examples, with the exception of (g), will be discussed only in relation to the Cartesian Grid form.

(b) \( \frac{\lambda}{2} \) and \( \frac{\lambda}{4} \) Transformers. A complete traverse of a \( u \) circle on the Cartesian Grid corresponds to an increment of \( n \) equal to \( 0.5 \), that is, to transformation of the load impedance through a \( \frac{\lambda}{2} \) length of loss-free line. This length of line clearly acts as a \( 1:1 \) transformer.

For a \( \frac{\lambda}{4} \) length of line \( n_0 = 0.25 \). Thus, if point \( P \) in Fig. 42 represents the point \( (r_T, x_T) \), the input impedance \( z_s \) of the \( \frac{\lambda}{4} \) line is obtained by proceeding round the \( u \) circle shown \( (u = u_1) \) until it intersects the arc \( n_1 + 0.25 \), which is the complementary of the arc \( n_1 \). Hence \( z_s z_T = 1 \) or \( Z_sZ_T = Z_{0}^2 \).

(c) The Short-circuited Line. \( Z_T = 0 \). In this case the input impedance of a length \( l \) of line is found by proceeding up the \( +jx \) axis (the circle \( u = u_1 = 0 \) from the origin, \( n = 0 \), to the arc \( n = \frac{l}{\lambda} \)). Near \( n = 0.25, \left( l = \frac{\lambda}{4} \right) \), \( z_s \) is very great and inductive, and when \( n \) passes the value \( 0.25 \), the impedance reappears, approaches the origin along the \( -jx \) axis (capacitive), and reaches the origin at

\[
n = 0.5 \left( l = \frac{\lambda}{2} \right).
\]

(d) Maximum and Minimum Values of Input Impedance. Progressive lengthening of a loss-free line causes the representative point for the input impedance to traverse the same \( u \) circle indefinitely. At the points where the circle cuts the \( r \) axis the impedance is purely resistive, and at intervals of \( n = 0.25, \left( l = \frac{\lambda}{4} \right) \), it passes successively through maximum and minimum values given by these points of intersection.

(e) Determination of \( Z_T \) from an Observed Standing Wave Pattern. When \( Z_T \) differs from \( Z_0 \) standing waves of voltage and current are present on the line. Suppose, as illustrated in Fig. 43 (a), that in a particular case the first maximum of voltage occurs at a point distant \( l \) from the
termination, and that the Standing Wave Ratio, \( \frac{V_{\text{max}}}{V_{\text{min}}} \), has the value \( \rho \). The input impedance of the length \( l \) of line terminated in \( Z_T \) is then purely resistive and of value \( \rho Z_0 \). Given this information \( Z_T \) may be determined in the manner illustrated in Fig. 43 (b). First locate on the \( r \) axis the point \( r = \frac{\rho Z_0}{Z_0} = \rho \); then move backwards through a distance \( n = \frac{l}{\lambda} \) along the \( u \) circle which intersects the \( r \) axis at this point. The point \( P \) reached gives \( Z_T = \frac{Z_T}{Z_0} \).

(f) Treatment of Admittances. If \( z_1 \) and \( z_2 \) are the points of intersection of a pair of complementary arcs with the same \( u \) circle, then, as mentioned under (b), \( z_1 z_2 = 1 \). Consequently \( z_2 = \frac{1}{z_1} \) and represents the admittance of \( z_1 \).

Now return to expression (4.17) for the input impedance \( Z_T \) and define the admittances \( Y_0 = \frac{i}{Z_0}, \ Y_T = \frac{i}{Z_T}, \ Y_T = \frac{i}{Z_0} \), \( Y_1 = \frac{i}{Z_1}, \ Y_2 = \frac{i}{Z_2} \). (4.17) can then be written

\[
y_T = y_0 + y_0 \tanh \frac{PL}{y_0 + y_T \tanh \frac{PL}{y_T}},
\]

and since this is formally the same as (4.17), the circle diagram may be used to solve it by the location of representation points \( y_T \) and \( y_0 \) in exactly the same manner as described above.

(g) Stub Matching.—Single Stub of Variable Position. The method of matching by means of a single stub is illustrated in Fig. 44 (a). This shows a feeder of characteristic impedance \( Z_0' \) and a load impedance \( Z_T \) which is to be transformed to provide a correct termination for this line. \( Z_T \) is connected to the feeder through an auxiliary line of length \( l_1 \), the characteristic impedance \( Z_0 \) of which need not be equal to \( Z_0' \), and a short-circuited stub of length \( l_2 \) is similarly connected. It being supposed that \( Z_0', Z_0 \) and \( Z_T \) are known, the problem is to determine the required line lengths \( l_1, l_2 \).

The problem is best handled in terms of admittances, and in these terms the requirement is that the combined
input admittance of the lines $l_1$ and $l_2$ shall be equal to $\frac{1}{Z_0}$. Now the input admittance of a short-circuited line, supposed loss-free, is a pure susceptance. Thus, if $l_1$ is chosen so that the input admittance of this line is equal to $\frac{1}{Z_0}$, the input admittance of the lines $l_1$ and $l_2$ shall be equal to $\frac{1}{Z_0}$. The range of impedance $Z_r$ which it is possible to match in this way is limited, but whether it is, or is not, possible in a given case is shown by the circle diagram. The procedure is as follows:

Form $y'_0 = \frac{Y_0}{Z_0}$ and $y_r = \frac{Y_r}{Z_0}$, and locate these points $S$ and $T$ on the circle diagram, Fig. 44 (b). In general $y'_0$ will be purely real and will therefore lie on the real axis.

Draw a perpendicular through $S$ and traverse the $u$ circle through $T$ until this perpendicular is encountered at $U$. This point corresponds to an admittance the real part of which is $y'_0 = \frac{1}{Z_0}$. The increment of $n$ involved in the movement from $T$ to $U$ gives the line length $l_1$ as $\Delta n \lambda$, while $US$ gives the magnitude of the susceptance to be provided by the line $l_2$. In order to obtain this length, mark off on the axis of imaginaries $OB = US$ in the reverse sense to $US$, Fig. 44 (c), and proceed round the semi-circle passing through $B$ to $C$. Note the value of the $n$ arc through $C$; $l_2$ is then given as $n\lambda$.

The same result may be obtained by use of the Polar form of diagram as follows. As before form $y'_0 = \frac{Y_0}{Z_0} = g'_0 + j\omega$, and $y_r = \frac{Y_r}{Z_0} = g_r + j\omega$, and locate these as the points $S$ and $T$ on the diagram. The former lies on the axis $x = 0$ and the latter at the intersection of the $r-g'$, $x = +\omega$, circles, as shown in Fig. 45 (a).

Through the point $T$, Fig. 45 (b), passes a particular $u$ circle and a radial $n$ line. This $u$ circle intersects the $r$ circle through $S$ at two points, $P$ and $P'$, both of which represent admittances having a real part equal to $y'_0$. Their respective imaginary parts $\pm \omega$ are given by the reactance circles which pass through these points. Thus the point $P$ corresponds to an admittance $y'_0 + j\omega$, the transformed value of $y_r$ through a length of line given by $(n_3 - n_1)\omega$, where $n_1$ and $n_2$ are the $n$ values of the radial lines $TR$ and $PQ$ through $T$ and $P$. This gives the desired value of $l_1$ in Fig. 44 (a).
The stub length $l_2$ follows from the susceptance value $-b$, which it is necessary to provide at the point of connexion. We shall consider a short-circuited stub for which

![Diagram](image)

the terminal admittance is infinite and is represented by the point $O$, Fig. 45 (b). The length of short-circuited stub having an input susceptance $-b$ is given by the displacement on the $n$ scale, reckoned towards the generator, of the point $W'$ from $O$. Thus the $n$ line through $O$ being

$0.25$ (Fig. 40), and that through $W'$, $n_3$ say, the desired length $l_2$ is given by $(n_3 - 0.25)\lambda$.

Similar considerations affecting the point $P'$, $y_0' - jb$, lead to an alternative matching arrangement.

(h) Two or Three Fixed Stubs of Adjustable Length. The minimum number of fixed stubs necessary to transform any given terminating impedance $Z_T$ to any desired $Z_0$ is three. On the other hand, matching can be accomplished over a considerable range of $Z_T$ by means of two stubs, and where this is possible the arrangement is naturally to be preferred. It is of importance, therefore, to ascertain the range of $Z_T$ over which the two-stub device will be adequate.

In considering the system of Fig. 46 (a) we shall assume for simplicity that the characteristic impedance of the matching line $TR$ is the same as that of the feeder to which it is joined at $R$. The normalized admittance to be provided just to the left of the second stub $Q$ is thus $1 + jo$.

Suppose, now, that the terminal admittance $y_T$ is represented on the Cartesian Grid by the point $T$ of Fig. 46 (b) and that when transformed through the length of line $TP$ it is represented by the point $S_1$. To this admittance any positive or negative value of susceptance can be added by suitable adjustment of the first stub, but let it be supposed that this is set arbitrarily at the positive value $S_1S_1'$. The point $S_1'$ thus gives the admittance at a point just to the left of the first stub.

The $u$ circle through $S_1'$ crosses the unit conductance line at $S_2$ and the line length $PQ$, Fig. 46 (a), necessary to effect this transformation is given by the difference in designation of the $n$ arcs passing through $S_1'$ and $S_2$. The addition of a susceptance $S_3S_3'$ at $Q$ now produces as resultant the desired unit conductance at this point.

Now, the point $S_3'$ may be made to occupy any position along the constant conductance line through $S_1$ by adjustment of the stub $P$, and, similarly, $S_3$ can be brought to $S_3'$ whatever position it occupies along the unit conductance
Moreover, consideration of the complete mesh of Fig. 38 will show that provided the point $S_1$ lies below the unit conductance line, the constant conductance line through it cuts all the $n$ arcs. Hence, whatever the line length $PQ$ between the stubs—excepting $\frac{\lambda}{2}$ or a multiple of it—two points, $S_2'$ and $S_3$, can always be found, which are separated by the corresponding value of $n$ ($\Delta n = \frac{PQ}{\lambda}$). If the length $PQ$ is very small or approaches $\frac{\lambda}{2}$, the shunt conductances required become inconveniently large for satisfactory control, but, subject to the above-mentioned proviso, the two-stub device will match with any arbitrary (fixed) spacing of the stubs.

When the point $S_1$ lies above the unit conductance line the constant conductance line cuts only a proportion of the $n$ arcs and the necessary conditions for matching can be satisfied for only a limited range of the stub separation $PQ$. This range decreases the farther $S_1$ lies above the unit conductance line.

The above discussion may have suggested that the stub separation $PQ$ is variable, whereas there are obvious practical reasons for fixing it. A separation of $\frac{\lambda}{8}$ or $\frac{3\lambda}{8}$ is frequently employed, and affords a good compromise between breadth of matching range and sensitivity of control.

The limitation of range can be overcome by providing for the insertion of a supplementary $\frac{\lambda}{4}$ length of line, when necessary, between the termination $T$, Fig. 46(a), and the first stub $P$. This serves to transform a point $S_1$ lying above the unit conductance line to one below it, and to enable a match to be effected on the two-stub device, as discussed above.

Alternatively, and preferably, the point $S_1$, corresponding
to the admittance just in front of stub $P$, can always be brought below this line by fitting a third stub between $T$ and $P$, as shown in Fig. 46 (c). This three-stub arrangement can be adjusted to match any terminating impedance. The separation of the stubs is arbitrary, excepting only that neither of the spacings must be $\frac{\lambda}{2}$, but again a frequently employed spacing is $\frac{\lambda}{8}$.

DETERMINATION OF THE PROPAGATION CONSTANTS OF FEEDER LINES BY USE OF THE CIRCLE DIAGRAM TECHNIQUE

The rationalized input impedances of short- and open-circuited lines of length $l$ are respectively

$$z_{s.c.} = \frac{Z_{s.c}}{Z_0} = \tanh (\alpha + j\beta)l ;$$

$$z_{o.c.} = \coth (\alpha + j\beta)l = \tanh \left\{ \alpha l + j \left( \frac{\pi}{2} + \beta l \right) \right\}$$

If, now, $l$ is varied over a range $\frac{\lambda}{2}$, so that $\beta l$ varies through an angle $\pi$, and it be supposed that $l$ embraces a sufficient number of wavelengths for $\alpha l$ to remain sensibly constant over the change, reference to equation (6.5), $(c = 1)$, shows that the locus of $z_{s.c.}$ and $z_{o.c.}$ is a circle of centre $(\coth 2\alpha /l, 0)$.

* It has been shown analytically by W. E. Willshaw and R. J. Clayton (General Electric Co.) that for a given separation $l$ between the stubs of the two-stub arrangement of Fig. 46 (a), the shunt input resistance $R$ at $P$ of the line $PT$ can be transformed to any value lying between $\frac{Z_0}{R} \sin^2 \frac{\pi}{\lambda} l$ and infinity, provided that $l$ is not zero or $\pi \frac{\lambda}{2}$. The range can clearly be increased by decreasing $Z_0$ and $l$. Additionally, if $R$ is small, thus limiting the range, an increase can be obtained, as mentioned above, by the addition of a supplementary quarter-wavelength line between $P$ and $T$.

† See page 114.
APPENDIX I

DATA FOR THE CONSTRUCTION OF A CARTESIAN GRID DIAGRAM (constant \( c = 1 \))

(a) The \( u \) Family of Circles

<table>
<thead>
<tr>
<th>Attenuation (decibels)</th>
<th>( 2u )</th>
<th>Centre ( r = \cosh 2u )</th>
<th>Radius ( \cosh 2u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.230</td>
<td>4.423</td>
<td>4.308</td>
</tr>
<tr>
<td>1.2</td>
<td>0.276</td>
<td>3.715</td>
<td>3.577</td>
</tr>
<tr>
<td>1.4</td>
<td>0.323</td>
<td>3.202</td>
<td>3.040</td>
</tr>
<tr>
<td>1.6</td>
<td>0.368</td>
<td>2.839</td>
<td>2.657</td>
</tr>
<tr>
<td>1.8</td>
<td>0.415</td>
<td>2.546</td>
<td>2.342</td>
</tr>
<tr>
<td>2.0</td>
<td>0.461</td>
<td>2.320</td>
<td>2.094</td>
</tr>
<tr>
<td>2.2</td>
<td>0.506</td>
<td>2.141</td>
<td>1.894</td>
</tr>
<tr>
<td>2.4</td>
<td>0.554</td>
<td>1.986</td>
<td>1.715</td>
</tr>
<tr>
<td>2.6</td>
<td>0.600</td>
<td>1.862</td>
<td>1.570</td>
</tr>
<tr>
<td>2.8</td>
<td>0.645</td>
<td>1.759</td>
<td>1.447</td>
</tr>
<tr>
<td>3.0</td>
<td>0.691</td>
<td>1.670</td>
<td>1.338</td>
</tr>
<tr>
<td>3.2</td>
<td>0.722</td>
<td>1.576</td>
<td>1.125</td>
</tr>
<tr>
<td>3.4</td>
<td>0.752</td>
<td>1.483</td>
<td>1.086</td>
</tr>
<tr>
<td>3.6</td>
<td>0.778</td>
<td>1.395</td>
<td>1.044</td>
</tr>
<tr>
<td>3.8</td>
<td>0.801</td>
<td>1.314</td>
<td>0.994</td>
</tr>
<tr>
<td>4.0</td>
<td>0.822</td>
<td>1.222</td>
<td>0.938</td>
</tr>
</tbody>
</table>

(b) The \( n \) Family of Arcs

All the \( n \) arcs, for constant \( c = 1 \), pass through the point \((1.0)\). Their centres along the \( x \) axis are given below. By drawing a complete semicircle for each calculated centre, two arcs of constants \( n \) and \((n + 0.25)\) are obtained.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Centre ( x = -\cot 4\pi n )</th>
<th>Centre ( x = -\cot 4\pi (n + 0.25) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>-7.916</td>
<td>+0.063</td>
</tr>
<tr>
<td>0.02</td>
<td>-3.895</td>
<td>+0.191</td>
</tr>
<tr>
<td>0.03</td>
<td>-2.526</td>
<td>+0.325</td>
</tr>
<tr>
<td>0.04</td>
<td>-1.819</td>
<td>+0.471</td>
</tr>
<tr>
<td>0.05</td>
<td>-1.376</td>
<td>+0.635</td>
</tr>
<tr>
<td>0.06</td>
<td>-1.065</td>
<td>+0.827</td>
</tr>
<tr>
<td>0.07</td>
<td>-0.827</td>
<td>+1.065</td>
</tr>
<tr>
<td>0.08</td>
<td>-0.635</td>
<td>+1.376</td>
</tr>
<tr>
<td>0.09</td>
<td>-0.471</td>
<td>+1.819</td>
</tr>
<tr>
<td>0.10</td>
<td>-0.325</td>
<td>+2.526</td>
</tr>
<tr>
<td>0.11</td>
<td>-0.191</td>
<td>+3.895</td>
</tr>
<tr>
<td>0.12</td>
<td>-0.063</td>
<td>+7.916</td>
</tr>
</tbody>
</table>
(a) The \( u \) Family of Circles

The centres of all these circles lie at the point 1.0 on the resistance \( \left( \frac{R}{Z_0} \right) \) axis of the diagram. The designations of the \( n \) lines which radiate from this point are given round the rim of the diagram in the manner shown in Fig. 40, one complete rotation representing a change of \( \left( = \frac{I}{\lambda} \right) \) equal to 0.5.

<table>
<thead>
<tr>
<th>Attenuation, db.</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius = ( e^{-2u} )</td>
<td>1.0</td>
<td>0.944</td>
<td>0.891</td>
<td>0.841</td>
<td>0.794</td>
<td>0.759</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Attenuation, db.</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius = ( e^{-2u} )</td>
<td>0.322</td>
<td>0.369</td>
<td>0.415</td>
<td>0.461</td>
<td>0.507</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Attenuation, db.</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
<th>3.0</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius = ( e^{-2u} )</td>
<td>0.553</td>
<td>0.600</td>
<td>0.646</td>
<td>0.692</td>
<td>0.807</td>
</tr>
</tbody>
</table>

(b) The \( r.x. \) Families of Circles

It should be noted that the \( r \) circles are all centred upon the resistance \( \left( \frac{R}{Z_0} \right) \) axis, and that all pass through the bottom point of the diagram, while the \( x \) circles, of which only arcs shown in Fig. 40, are centred along a line passing through this point and normal to the resistive axis. All these arcs also pass through the bottom point of the diagram.

<table>
<thead>
<tr>
<th>The ( r ) Circles</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = \frac{R}{Z_0} )</td>
</tr>
<tr>
<td>Radius = ( \frac{I}{r+1} )</td>
</tr>
</tbody>
</table>

| \( r = \frac{R}{Z_0} \) | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 | 0.55 |
| Radius = \( \frac{I}{r+1} \) | 0.769 | 0.741 | 0.714 | 0.689 | 0.667 | 0.645 |

| \( r = \frac{R}{Z_0} \) | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.90 |
| Radius = \( \frac{I}{r+1} \) | 0.625 | 0.606 | 0.588 | 0.571 | 0.556 | 0.526 |

| \( r = \frac{R}{Z_0} \) | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 |
| Radius = \( \frac{I}{r+1} \) | 0.5 | 0.476 | 0.454 | 0.435 | 0.417 | 0.400 |

| \( r = \frac{R}{Z_0} \) | 1.6 | 1.8 | 2.0 | 2.2 | 2.4 | 2.6 |
| Radius = \( \frac{I}{r+1} \) | 0.385 | 0.357 | 0.333 | 0.313 | 0.294 | 0.278 |

| \( r = \frac{R}{Z_0} \) | 3.0 | 3.5 | 4.0 | 6.0 | 10.0 | 20.0 |
| Radius = \( \frac{I}{r+1} \) | 0.250 | 0.222 | 0.200 | 0.143 | 0.091 | 0.047 |
The $x$ Circles

<table>
<thead>
<tr>
<th>$x = \frac{X}{Z_0}$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius = $\frac{1}{x}$</td>
<td>$\infty$</td>
<td>1.0</td>
<td>5.0</td>
<td>3.33</td>
<td>2.5</td>
<td>2.0</td>
<td>1.67</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x = \frac{X}{Z_0}$</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius = $\frac{1}{x}$</td>
<td>1.43</td>
<td>1.25</td>
<td>1.11</td>
<td>1.0</td>
<td>0.833</td>
<td>0.714</td>
<td>0.625</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x = \frac{X}{Z_0}$</th>
<th>1.8</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius = $\frac{1}{x}$</td>
<td>0.55</td>
<td>0.50</td>
<td>0.40</td>
<td>0.333</td>
<td>0.286</td>
<td>0.23</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Any desired number of circles additional to those designated above may of course be incorporated, depending on the size of the diagram and the degree of accuracy desired.

**SUPPLEMENTARY BIBLIOGRAPHY**

### VALVE OSCILLATORS INCORPORATING TRANSMISSION LINE ELEMENTS


### LINE RESONATORS AS FREQUENCY STABILIZERS


### BOOKS INCORPORATING A TREATMENT OF TRANSMISSION LINES IN RELATION TO VERY HIGH FREQUENCIES


INDEX

Admittance of lines—circle diagram treatment, 134

Attenuation constant, 32, 49
coaxial line, 50
condition for minimum value, 51
measurement, 114, 142
twin lines, 52

Basic equations for transmission lines, 22

Capacitance per unit length:
coaxial line, 41
twin lines, 43, 44

Characteristic impedance, 32, 45
effect of small changes of dimensions, 47
measurement, 114, 142

Circle diagrams:
Cartesian Grid form, 121
Polar form, 125

Conductance—dielectric, 39
Conductivity of metals, 36

Decibel, 49
Depth of current penetration, 36

Discontinuities—effects of, 29, 62, 66

Eccentricity in coaxial lines, 43, 47

Efficiency of transmission, 87
Electric stress in coaxial line, 53

Feeders, 1
Filters—line, 116
Frequency stabilizers, 13

Half-wave transformer, 119, 132

High frequency resistance, 34

Impedance:
characteristic, 32, 45
determination by circle diagram, 133
measurement by 'Brückmann' method, 81
measurement by 'Chipman' method, 109
measurement by other methods, 112
transformation, 6, 119
wave, 63

Inductance per unit length:
coaxial line, 41
twin lines, 43, 44

Input impedance:
general expression, 81, 119
of S.C. and O.C. lines, 93, 95

Low-loss reactances, 95

Matching:
at generator, 2, 5, 86
at load, 2, 6, 59, 69, 74, 134, 139

Maxwell's equations, 16

M.K.S. units, 18

Neper, 49

Open-ended line, 68

Permittivity, 18
measurement, 112
relative to air, 40

Phase constant, 32, 47
measurement, 114, 142
<table>
<thead>
<tr>
<th>Topic</th>
<th>Page(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power factor of dielectric</td>
<td>40</td>
</tr>
<tr>
<td>Measurement</td>
<td>102</td>
</tr>
<tr>
<td>Power measurement from standing wave pattern</td>
<td>83</td>
</tr>
<tr>
<td>Power rating of lines:</td>
<td>85</td>
</tr>
<tr>
<td>- on electric stress basis</td>
<td>53</td>
</tr>
<tr>
<td>- on thermal basis</td>
<td>55</td>
</tr>
<tr>
<td>Power transfer</td>
<td>86</td>
</tr>
<tr>
<td>Propagation constant</td>
<td>32</td>
</tr>
<tr>
<td>Q. value of resonant lines</td>
<td>99</td>
</tr>
<tr>
<td>Measurement</td>
<td>105</td>
</tr>
<tr>
<td>Quarter-wave transformer</td>
<td>3, 134</td>
</tr>
<tr>
<td>Range of non-propagated modes</td>
<td>20, 67</td>
</tr>
<tr>
<td>Reflection coefficients</td>
<td>60, 65</td>
</tr>
<tr>
<td>- determination from standing wave pattern</td>
<td>78</td>
</tr>
<tr>
<td>- particular values</td>
<td>76</td>
</tr>
<tr>
<td>Resistance per unit length</td>
<td>41</td>
</tr>
<tr>
<td>- coaxial line</td>
<td>41</td>
</tr>
<tr>
<td>- twin lines</td>
<td>43, 44</td>
</tr>
<tr>
<td>Resistive disk termination</td>
<td>69</td>
</tr>
<tr>
<td>Resonant lengths of lines terminated arbitrarily</td>
<td>106</td>
</tr>
<tr>
<td>Resonant lines</td>
<td>90</td>
</tr>
<tr>
<td>Short-circuited line</td>
<td>65</td>
</tr>
<tr>
<td>Standing wave detector</td>
<td>80</td>
</tr>
<tr>
<td>- ratio, 2, 71, 76</td>
<td></td>
</tr>
<tr>
<td>- representation of voltage</td>
<td>75</td>
</tr>
<tr>
<td>Stub matching</td>
<td>7</td>
</tr>
<tr>
<td>- single stub, 7, 134</td>
<td></td>
</tr>
<tr>
<td>- two and three stubs</td>
<td>8, 139</td>
</tr>
<tr>
<td>Tapped lines</td>
<td>97</td>
</tr>
<tr>
<td>Terminated lines</td>
<td>59</td>
</tr>
<tr>
<td>- correctly terminated</td>
<td>59, 69, 74</td>
</tr>
<tr>
<td>- determination of impedance value to represent behaviour</td>
<td>81</td>
</tr>
<tr>
<td>- reflection coefficients, 60, 65</td>
<td></td>
</tr>
<tr>
<td>- resonant lengths, 106</td>
<td></td>
</tr>
<tr>
<td>- short circuited, 65</td>
<td></td>
</tr>
<tr>
<td>Transformers—impedance</td>
<td>3, 119</td>
</tr>
<tr>
<td>Valve oscillators</td>
<td>9</td>
</tr>
</tbody>
</table>